Portfolio Choice and Ambiguous Background Risk*

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October 30, 2014

Abstract

This paper examines how an ambiguous background risk influences portfolio choice. We first establish conditions for ranking the expected-utility choices under each of the competing prior distributions of the background risk. We then use both the smooth ambiguity model of Klibanoff, Marinacci and Mukerji (2005) and the maxmin expected utility model of Gilboa and Schmeidler (1989) to show how ambiguity aversion lessens the demand for the risky asset. The implications of this result for the equity premium puzzle are also examined.

JEL classification numbers: D81, G11

Keywords: Ambiguity, Ambiguity aversion, Background risk, Monotone comparative statics, Portfolio choice

*The authors thank Aurelien Baillon, Christian Gollier and Jens Jackwerth for helpful comments on a draft of this paper.

1 Introduction

Much attention has been paid to how economic and financial decisions are affected by exogenous “background risks” that cannot be traded by economic agents. This introduces a type of incompleteness into the market. Even in cases where the background risk is independent from endogenous risks, it can affect behavior towards the endogenous risk; see for example, Doherty and Schlesinger (1983) and Gollier and Pratt (1996). The latter authors look at restrictions on expected-utility preferences under which a non-positive mean, independent background risk causes a more cautious behavior on the part of the economic agent, such as investing less in risky assets or purchasing more insurance. In particular: when does such an independent background risk cause the agent to behave "as if" she was more risk averse? A related question was posed by Eeckhoudt, Gollier and Schlesinger (EGS, 1996), who ask: under what restrictions on expected-utility preferences will a deterioration in the background risk – by either first-order or second order stochastic dominance – cause the agent to act "as if" she is more risk averse?

In this paper, we consider a model in which there are several potential distributions for the independent, exogenous background risk. The economic agent is assumed to form a subjective set of probabilities over the competing prior distributions of the background risk. In this sense, the background risk is assumed to be ambiguous. Using the smooth ambiguity aversion model of Klibanoff, Marinacci and Mukerji (KMM, 2005), we ask whether or not ambiguity aversion will cause the agent to behave in a more risk averse manner towards the (non-ambiguous) endogenous risk. We also show how the result applies in the case where preferences exhibit Gilboa and Schmeidler's (1989) maxmin expected utility.

For the case of concreteness, we examine the standard portfolio problem in which the investor must allocate her wealth between one risky and one riskless asset. However, the result is easily applied in other settings where "more risk-averse behavior" leads to known consequences, such as models of insurance purchasing. In our model, it is important to note that the distribution of returns on the risky asset is assumed to be non-ambiguous. Only the independent background risk is ambiguous. The investor has her own (subjective) belief about asset returns, but she is quite unsure about her own background risk. She has her own subjective beliefs about the likelihoods of potential distributions of background risk. The investor's final wealth thus consists of three components: (1) a safe asset, (2) a risky asset, and (3) an exogenous and ambiguous background risk.

We determine conditions under which the ambiguity-averse investor decides on less risky asset holdings when the background risk is ambiguous, as compared to the case of a non-ambiguous background risk. The smooth ambiguity model of KMM (2005) captures ambiguity attitude via a second-order utility function over the competing values of expected utility associated with each prior distribution of background risk. We need to impose conditions on both the first-order and second-order utility to obtain definitive qualitative results. The condition required on the first-order utility is due to EGS (1996) and allows us to rank the investments made by an expected-utility maximizer under each of the competing prior distributions for the background risk. This allows us to apply a comparative static method introduced by Jewitt (1987) and Athey (2002) to isolate and analyze the effect of ambiguity aversion on portfolio choice.

Our model is complementary to the one by Gollier (2011), who examines a model without background risk, but with ambiguous asset returns. In our model, the asset returns are risky, but are not ambiguous. The model presented here is also similar in spirit to those by Baillon (2013) and Berger (2014). These authors examine the effects of an ambiguous future-income risk on the demand for precautionary saving, in a world without risky asset returns.

The next section provides a setting for the general model. Section 3 extends the results of EGS (1996) to conditions under which ranking of the competing distributions of the background risk via N^{th} order stochastic dominance allows us also to rank the levels invested in the risky asset. Section 4 presents our main result, showing that ambiguity aversion leads to a decreased investment in the risky asset, when the background risk is ambiguous. The following section shows the implication of our results for the equity premium puzzle, before providing some concluding remarks.

2 The Model

Consider the standard portfolio problem in which a risk-averse investor chooses between two assets: one is a safe asset and the other is a risky asset. The safe asset has a net return of r_f, which we normalize to zero. The risky asset has the random net return $\widetilde{x}$ with distribution function F defined over the bounded support $[-1, \overline{x}]$. The expected return $E\widetilde{x}$ is assumed to be strictly positive, which guarantees that the optimal holding of the risky asset is also strictly positive under expected utility. The investor is endowed with the certain initial wealth $w > 0$.

If we introduce an additive, statistically independent background risk into the model, the optimal investment in the risky asset will likely change. Let $\tilde{\epsilon}$ denote this background risk with distribution function G defined over the bounded support $[a, b]$. We assume that this background risk is disliked, with a non-positive mean, $E[\tilde{\epsilon}] \leq 0$. The investor maximizes her expected utility. The utility function u is assumed to be increasing and concave. We also assume that $u^k \equiv d^k u/dw^k$ exists and is continuous for k as needed. The optimal amount invested in the risky asset is given by [1]

$$\alpha_0 \equiv \arg\max_{\alpha} E\left[u\left(w + \alpha\tilde{x} + \tilde{\epsilon}\right)\right]. \tag{1}$$

Conditions on preferences under which the presence of such a background risk will cause a decrease in α, a property known as "risk vulnerability," are examined in detail by Gollier and Pratt (1996). It is straightforward to

[1] The second–order condition is satisfied by risk aversion, which also guarantees that expected utility is concave in α. Thus, the optimal portfolio is unique.

show that $\alpha_0 > 0$ when $E\widetilde{x} > 0$.

Our main objective in this paper is to consider what happens when the background risk $\tilde{\epsilon}$ is ambiguous. To this end, we suppose that there are $n \geq 2$ possible distributions $\{G_1, G_2, \ldots G_n\}$ for background risk $\tilde{\epsilon}$. We denote $\tilde{\epsilon}_\theta$ as the background risk if distribution function G_θ is the true distribution function. The investor has subjective probability q_θ for the likelihood of $\tilde{\epsilon}_\theta$ being the true background risk. The distribution G is assumed to be equal to $G = \sum_\theta q_\theta G_\theta$. In other words, probability beliefs for $\tilde{\epsilon}$ are the subjectively weighted distribution of the priors.

Given each background risk $\tilde{\epsilon}_\theta$, the investor computes that her expected utility is $E\left[u\left(w + \alpha\tilde{x} + \tilde{\epsilon}_\theta\right)\right]$. In this analysis, it is assumed that the investor follows the smooth ambiguity model introduced by KMM (2005). By introducing the second–order utility ϕ over the (first–order) utility for wealth, the investor's utility becomes

$$\sum_{\theta=1}^{n} q_\theta \phi\left(E\left[u\left(w + \alpha\tilde{x} + \tilde{\epsilon}_\theta\right)\right]\right). \tag{2}$$

ϕ is assumed to be increasing and twice differentiable.

The function ϕ captures the investor's attitudes toward ambiguity. A linear ϕ degenerates back to expected-utility preferences. In this case, the ambiguity has no influence on portfolio decisions and the optimal investment in the risky asset is identical to that without ambiguity, α_0. A concave ϕ connotes that the investor is ambiguity averse. Such an investor is made worse off by the fact that the background risk is ambiguous; but such ambiguity does not necessarily lead to a reduced investment in the risky asset.

The investor chooses her optimal portfolio to maximize total utility given as (2). The first–order condition is[2]

$$\sum_{\theta=1}^{n} q_\theta \phi' \left(E\left[u\left(w + \alpha^* \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \right) E\left[\tilde{x} u' \left(w + \alpha^* \tilde{x} + \tilde{\epsilon}_\theta \right) \right] = 0 \qquad (3)$$

The purpose of this paper is to determine how ambiguity aversion affects the optimal portfolio choice. In particular, we wish to know whether ambiguity aversion leads the individual to behave more cautiously and invest less in the risky asset, $\alpha^* \leq \alpha_0$. Due to the concavity of the objective function in α (since both u and ϕ are concave), this condition will hold whenever

$$\sum_{\theta=1}^{n} q_\theta \phi' \left(E\left[u\left(w + \alpha_0 \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \right) E\left[\tilde{x} u' \left(w + \alpha_0 \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \leq 0 \qquad (4)$$

3 Preliminary Results

To obtain concrete results, we consider the situation in which the prior distributions of the background risk can be ranked via the N^{th} order stochastic dominance (NSD). In particular, we assume that $\tilde{\epsilon}_j$ dominates $\tilde{\epsilon}_i$ in the sense of NSD for all i, j with $i < j$. Before considering the main question of how ambiguity affects the optimal portfolio, we first require the following result that allows us compare the optimal choices that would be made using expected utility, if the true distribution of the background risk (G_θ) was known. This result is adapted from EGS (1996).

The derived utility function, introduced by Kihlstrom *et al.* (1981) and

[2] The second–order condition is trivially satisfied when both u and ϕ are concave, with one of them strictly concave.

Nachman (1982), is defined as

$$v_\theta (w) \equiv E \left[u \left(w + \tilde{\epsilon}_\theta \right) \right]. \tag{5}$$

We let $A(w)$ and $P(w)$ denote the measures of (absolute) risk aversion in the sense of Pratt (1964) and absolute prudence in the sense of Kimball (1990) respectively. That is, $A_u(w) \equiv -u''(w)/u'(w)$ and $P_u(w) \equiv -u'''(w)/u''(w)$.

Theorem 1 (Eeckhoudt, Gollier and Schlesinger, 1996 - extension): *Let $\tilde{\epsilon}_j$ dominate $\tilde{\epsilon}_i$ in the sense of N^{th} order stochastic dominance. Then $A_{v_i}(w) \equiv -v_i''(w)/v_i'(w) \geq -v_j''(w)/v_j'(w) \equiv A_{v_j}(w)$ $\forall w$ if and only if there exists a positive scalar λ_k such that*

$$-\frac{u^{k+2}(w+\epsilon)}{u^{k+1}(w+\epsilon)} \geq \lambda_k \geq A_u\left(w+\epsilon'\right) \tag{6}$$

for all $\epsilon, \epsilon' \in [a, b]$ and for all $k = 1, 2, \ldots, N$.

A formal proof for the above appears in EGS (1996) for $N = 1$ and $N = 2$. But a careful examination of their appendix proof shows that it holds for any arbitrary N with very straightforward extensions. In the case where $N = 1$, the necessary and sufficient condition in the Theorem reduces to $P_u(w+\epsilon) \geq \lambda_1 \geq A_u(w+\epsilon')$, which is the condition for decreasing risk aversion in the sense of Ross (1981). If $N = 2$, we need to add the condition that $-\frac{u^4(w+\epsilon)}{u^3(w+\epsilon)} \geq \lambda_2 \geq A_u(w+\epsilon')$. This condition guarantees that any mean-preserving increase in the background risk in the sense of Rothschild and Stiglitz (1970) will cause the derived utility function to become more risk averse.[3]

From the above Theorem, it is straightforward to order the amounts invested in the risky asset under alternative distributions for $\tilde{\epsilon}_\theta$. Let α_θ denote the optimal investment in the risky asset for an expected utility maximizer facing the (unambiguous) background risk $\tilde{\epsilon}_\theta$ for $\theta = 1, 2, ..., N$. In particular, we obtain the following result.

Corollary 1: *Assume that background risk $\tilde{\epsilon}_j$ dominates $\tilde{\epsilon}_i$ in the sense of the NSD for all $i < j$, $i = 1, 2, ..., n-1$, and that the investor faces one of these background risks. Further assume that condition* (6) *holds. Then the optimal investments in the risky asset satisfy $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.*

4 Ambiguous Background Risk

We now turn to the case where the background risk is ambiguous. We maintain the assumption that $\tilde{\epsilon}_j$ dominates $\tilde{\epsilon}_i$ in the sense of the NSD for all i, j with $i < j$. Since the distribution of $\tilde{\epsilon}$ was defined as $G = \sum_\theta q_\theta G_\theta$, it follows trivially that $\tilde{\epsilon}$ dominates $\tilde{\epsilon}_1$ in the sense of NSD, and that $\tilde{\epsilon}_n$ dominates $\tilde{\epsilon}$ in the sense of NSD. Assuming that (6) holds, it follows from Corollary 1 that $\alpha_1 \leq \alpha_0 \leq \alpha_n$. That is, the expected utility maximizer facing the background risk $\tilde{\epsilon}$ would choose a level of investment in the risky

[3] We note here that satisfying condition (6) is not trivial. Eeckhoudt et al. (1996) provide a few examples on a limited wealth domain. The condition $P_u(w+\epsilon) \geq \lambda_1 \geq A_u(w+\epsilon')$ together with $-\frac{u^4(w+\epsilon)}{u^3(w+\epsilon)} \geq \lambda_2 \geq A_u(w+\epsilon')$ is weaker that assuming both decreasing Ross risk aversion and decreasing Ross prudence, $-\frac{u^4(w+\epsilon)}{u^3(w+\epsilon)} \geq \lambda_2 \geq P_u(w+\epsilon')$. Indeed, as shown by Eeckhoudt et al. (1996), decreasing prudence in the sense of Kimball (1990) together with decreasing Ross risk aversion is sufficient when $N = 2$. Denuit et al. (2013) show that any utility function satisfying these conditions on an unrestricted domain must belong to the "Linex" class as introduced by Bell (1988).

asset α_0 that falls somewhere between those chosen under the worst prior distribution $\tilde{\epsilon}_1$ and the best prior distribution $\tilde{\epsilon}_n$.

It thus follows that there exists some j, $2 \leq j \leq n-1$, such that $\alpha_1 \leq ... \leq \alpha_j \leq \alpha_0 \leq \alpha_{j+1} \leq ... \leq \alpha_n$. Let θ_0 be any real number between j and $j+1$. Define the function $h(\theta) \equiv E\left[\tilde{x}u'\left(w + \alpha_0\tilde{x} + \tilde{\epsilon}_\theta\right)\right]$.[4]

From Corollary 1 and from the observation above, it follows that $h(\theta)$ is negative for all θ with $\alpha_\theta < \alpha_0$, and that $h(\theta)$ is positive for all θ with $\alpha_\theta > \alpha_0$. In other words, $h(\theta)$ satisfies the *single crossing property from below*; i.e.

$$(\theta - \theta_0)h(\theta) \geq 0, \quad \forall\theta = 1, 2, ..., n. \tag{7}$$

Before proceeding, we require the following Lemma, which follows from a comparative static technique introduced by Jewitt (1987) and Athey (2002) in the economic literature on uncertainty, as summarized in Gollier (2001, Proposition 16). A function $\psi(\theta, \delta)$ is defined to be *log-supermodular* if $\psi(\theta_H, \delta_L)\,\psi(\theta_L, \delta_H) \leq \psi(\theta_L, \delta_L)\,\psi(\theta_H, \delta_H)$ for every $\theta_L < \theta_H$ and $\delta_L < \delta_H$.

Lemma 1: *Let h satisfy the single crossing property from below, i.e.* (7). *Then*

$$E\left[h\left(\tilde{\theta}\right)\psi\left(\tilde{\theta}, \delta_H\right)\right] = 0 \Rightarrow E\left[h\left(\tilde{\theta}\right)\psi\left(\tilde{\theta}, \delta_L\right)\right] \leq 0, \ \forall\delta_L \geq \delta_H \tag{8}$$

if and only if ψ is log-supermodular. Moreover, if h does not satisfy (7)*,*

[4] Since α_0 is optimal when the background risk is $\tilde{\epsilon}$ under expected utility, note that $E\left[\tilde{x}u'\left(w + \alpha_0\tilde{x} + \tilde{\epsilon}\right)\right] = 0$

there exists a log-supermodular function ψ violating (8)

Note that with ambiguous background risk, the ambiguity-neutral investor would choose the same optimal investment in the risky asset as the expected utility maximizer, α_0. We are now able to prove our main result, that the ambiguity averse investor would choose a lower optimal investment in the risky asset, $\alpha^{*}<\alpha_0$.

Theorem 2: *Consider an investor who faces the ambiguous background risk $\tilde{\epsilon}_\theta$, $\theta = 1, ..., n$. Let $\tilde{\epsilon}_j$ dominate $\tilde{\epsilon}_i$ in the sense of N^{th} order stochastic dominance for all $i < j$ and assume that first-order utility satisfies condition* (6). *An ambiguity averse investor (with $\phi'' < 0$) will invest less in the risky asset than an ambiguity-neutral investor with identical first-order utility u and identical subjective probability beliefs q_θ.*

Proof: Since the function $h(\theta) = E\left[\tilde{x}u'\left(w + \alpha_0\tilde{x} + \tilde{\epsilon}_\theta\right)\right]$ satisfies the single crossing property (7), we can adapt standard expected utility arguments (see, for example, Gollier 2001, Lemma 2) to apply to second-order utility. In particular, let $\phi(u,i)$ denote $\phi_i(u)$ for $i = 1, 2$ Then $\phi'(u,i) \equiv \partial\phi(u,i)/\partial u$ is log-supermodular if and only if ϕ_1 is more ambiguity averse than ϕ_2. If we assume that ϕ_2 is ambiguity neutral, then $\phi'(\cdot,i)$ is log-supermodular if and only if ϕ_1 is ambiguity averse.

By Lemma 1, ϕ_1 is ambiguity averse if and only if

$$\begin{array}{c} \sum_{\theta=1}^{n} q_\theta h(\theta) = 0 \\ \Rightarrow \sum_{\theta=1}^{n} q_\theta \phi_1' \left(E\left[u\left(w + \alpha_0 \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \right) h(\theta) \leq 0 \end{array} . \tag{9}$$

The equality in (9) above is due to the fact that the optimal portfolio under the ambiguity-neutral ϕ_2 is equal to α_0. This inequality, which can be rewritten as $\sum_{\theta=1}^{n} q_\theta \phi_1' \left(E\left[u\left(w + \alpha_0 \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \right) E\left[\tilde{x} u'\left(w + \alpha_0 \tilde{x} + \tilde{\epsilon}_\theta \right) \right] \leq 0$, shows that the ambiguous background risk decreases the optimal level of the risky asset; i.e. $\alpha^* \leq \alpha_0$. ■

Applying the second part of Lemma 1, we see that ambiguity aversion alone does not guarantee that ambiguous background risk decreases the optimal portfolio of risk asset if there are no restrictions on the possible distributions and on the first-order utility. In such cases, $h(\theta)$ may violate the single crossing condition.

It also is interesting to note from Corollary 1, that Theorem 2 will also hold for the max-min expected utility model of Gilboa and Schmeidler (1989) as a limiting case.[5] Assuming that first order utility satisfies (6), an investor with these preference will make the investment choice of α_1, which is less than the expected utility level of investment in the risky asset α_0. We state this formally as Corollary 2.

Corollary 2: *Consider an investor who faces the ambiguous background risk $\tilde{\epsilon}_\theta$, $\theta = 1, ..., n$. Let $\tilde{\epsilon}_j$ dominate $\tilde{\epsilon}_i$ in the sense of N^{th} order stochastic*

[5] See Proposition 3 in KMM (2005).

dominance for all $i < j$ and assume that first-order utility satisfies condition (6). *An investor with Gilboa and Schmiedler* (1989) *maxmin ambiguity preferences will invest less in the risky asset than an ambiguity-neutral investor with identical first-order utility u and identical subjective probability beliefs q_θ.*

Although the condition (6) might be hard to verify, it is less difficult for low values of N. For example, it follows from Theorem 2 that if the $\tilde{\epsilon}_\theta$ can be ranked via first-order stochastic dominance (FSD) and first-order utility satisfies decreasing absolute risk aversion in the strong sense of Ross (1981), then ambiguity aversion lowers the investment in the risky asset. If the distributions for the $\tilde{\epsilon}_\theta$ are continuous and also satisfy the monotone likelihood ratio property (MLRP), which is a stronger condition than FSD, then simple decreasing absolute risk aversion (DARA) in the sense of Arrow (1971) and Pratt (1964) can replace the condition (6). We show this formally in Corollary 3.

Corollary 3: *Assume that background risk $\tilde{\epsilon}_j$ dominates $\tilde{\epsilon}_i$ in the sense of the MLRP for all $i < j$, $i = 1, 2, ..., n-1$. Further assume that the first-order utility exhibits decreasing absolute risk aversion* (*DARA*). *Then, ceteris paribus, the optimal investment in the risky asset will be lower under ambiguity aversion than under ambiguity neutrality.*

Proof: Let g_θ denote the density function for G_θ. Define $\hat{g}_\theta(\epsilon) \equiv$

$g_\theta(\epsilon)\left[u'(w+\epsilon)/Eu'(w+\tilde{\epsilon}_\theta)\right]$. Since $\hat{g}_\theta$ is positive and $\hat{G}_\theta(b) = 1$, $\hat{G}_\theta$ can be viewed as distribution function. Using $\hat{G}_\theta$, it is straightforward to show that the degree of absolute risk aversion for the derived utility function (5) can be written as

$$A_v(w) = \int_a^b A_u(w+\epsilon)\, d\hat{G}_\theta. \tag{10}$$

By the construction of $\hat{G}_\theta$, $\hat{G}_j$ dominates $\hat{G}_i$ in the sense of the MLRP if and only if G_j dominates G_i in the sense of the MLRP, which is a stronger condition than FSD. If $A_u(x+\epsilon)$ is a decreasing function, then

$$A_{v_i}(w) = \int_a^b A_u(w+\epsilon)\, d\hat{G}_i \geq \int_a^b A_u(w+\epsilon)\, d\hat{G}_j = A_{v_i}(w). \tag{11}$$

It follows from (11) that we can rank the optimal investment choices for each $\tilde{\epsilon}_\theta$ as: $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$.

The rest of the proof mimics that of Theorem 2 and is omitted. ■

5 Equity premium puzzle

We consider here the implication of our result for the equity premium puzzle, initially posed by Mehra and Prescott (1985). The above analysis of the portfolio choice can also be applied to that of Lucas' (1978) model of asset pricing. The Lucas model with a background risk was examined in an expected-utilty framework by Gollier and Schlesinger (2002). We can easily extend their model to include ambiguity aversion, as modeled above.

A representative agent (investor) is assumed to be endowed with one unit of the risky asset. The price is of the risky asset is denoted P and any additional demand for the risky asset is denoted by β. Given this setting, the ambiguity-averse representative agent maximizes her total utility:

$$\max_{\beta} \sum_{\theta=1}^{n} q_{\theta} \phi \left(E \left[u \left(\tilde{y}_{\theta} \right) \right] \right), \text{ where } \tilde{y}_{\theta} = w + \tilde{x} + \tilde{\epsilon}_{\theta} + \beta \left(\tilde{x} - P \right). \tag{12}$$

In equilibrium, we must have $\beta = 0$, so that the first-order condition satisfies

$$\sum_{\theta=1}^{n} q_{\theta} \phi' \left(E \left[u \left(w + \tilde{x} + \tilde{\epsilon}_{\theta} \right) \right] \right) E \left[\left(\tilde{x} - P^{*} \right) u \left(w + \tilde{x} + \tilde{\epsilon}_{\theta} \right) \right] = 0. \tag{13}$$

In (13) above, P^{*} denotes the equilibrium market price for the risk asset in this economy, where the investor faces an ambiguous background risk.

Let P_0 denote the equilibrium asset price in an economy with an ambiguous background risk, but with ambiguity neutrality for the representative agent. Note that P_0 is also the equilibrium asset price in an economy in which the background risk is unambiguous.

We now pose the question of whether or not ambiguity aversion will decrease the equilibrium asset price. This effect is equivalent to increasing the equity premium. Analogous to Gollier and Schlesinger (2002), we examine first-order derivatives when in equilibrium, with $\beta = 0$. In particular, we wish to determine if

$$\begin{aligned} &\sum_{\theta=1}^{n} q_\theta \left(E\left[(\tilde{x} - P_0)\, u\left(\widetilde{w}_\theta\right)\right] - E\left[(\tilde{x} - P_0)\, u\left(\widetilde{w}_\theta\right)\right] \right) = 0 \\ \Longrightarrow &\sum_{\theta=1}^{n} q_\theta \phi'\left(E\left[u\left(\widetilde{w}_\theta\right)\right]\right) \left(E\left[(\tilde{x} - P_0)\, u\left(\widetilde{w}_\theta\right)\right] - E\left[(\tilde{x} - P_0)\, u\left(\widetilde{w}_\theta\right)\right] \right) \leq 0, \end{aligned} \tag{14}$$

where $\widetilde{w}_\theta \equiv w + \tilde{x} + \tilde{\epsilon}_\theta$

Given the assumptions in either Theorem 2 or in Corollary 2, it follows easily that (14) holds. At price P_0, the ambiguity averse representative agent demands too little of the risky asset. Thus, the equilibrium price P^* must be less than P_0.

It has been argued that an unambiguous background risk alone is not enough to explain the equity premium puzzle, *e.g.* Telmer (2003) and Lucas (2004). Adding ambiguity aversion leads to a higher equilibrium equity premium, thus (at least partly) explaining the higher observed equity premium.

6 Conclusion

This paper examines conditions under which ambiguity of a background risk leads to more cautious behavior by a risk averse investor. We first extend the EGS (1996) conditions on the first-order utility u in order to rank the levels of investment in the risky asset, when the competing prior distributions for the background risk are themselves ranked via N^{th} order stochastic dominance. Using both the smooth ambiguity model of KMM (2005) and the maxmin expected utility model of Gilboa and Schmeidler

(1989), we then show how ambiguity aversion reduces the investment in the risky asset vis-a-vis a non-ambiguous background risk. We also discuss the implications of this finding on the equity premium puzzle.

Together with the papers from Gollier (2011) and Berger (2014), a clear picture is emerging. In both of these papers, the authors consider a single risk – either a risky market asset or a risky future income – and assume that the risk is ambiguous in the sense of having multiple priors. Assuming that the background-risk priors can be ranked via first-order or second-order stochastic dominance, these authors show how ambiguity aversion overweights the "worse" prior distributions of the risk. Whereas the above models consider only a single source of risk, we consider a model with two risks, where one risk is ambiguous. In our model, we have an endogenous market risk, but we assume that there is an exogenous background risk that is ambiguous. Once again, if the priors can be ranked via stochastic dominance, we show how the ambiguity-averse individual overweights the "worse" priors. We do not restrict ourselves to only first- and second-order stochastic dominance changes and we allow for ranking by stochastic dominance of any arbitrary degree N.

The notion that background risks might be more ambiguous than other risks, such as market risk, seems reasonable to us. There is much data for asset returns and/or for other types of risk that are traded on the market. That is not to say that such market risks are not themselves subject to ambiguity, such as in Gollier (2011) or Berger (2014), but it would seem that background risk, which might include such risks as human-capital risk or the risk of personal health care costs, is especially prone to "guessing" about a probability distribution.

Of course, a more realistic model would allow for both types of ambiguity, including perhaps ambiguous correlations between the risks. Also, our model here examines and additive background risk. But other types of background risk might be multiplicative in nature.[6] Moreover, in a world with multivariate preferences, where non-wealth attributes such as health or environmental quality might matter, the background risk might be manifest in one of these non-wealth dimensions. All of these models seem open to the type of investigation undertaken in this current paper. Hopefully, our analysis is a useful framework for these situations.

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