# WHEN ROSS MEETS BELL: THE LINEX UTILITY FUNCTION

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**Abstract**

At first glance, there would appear to be no relationship between Bell's (1988) concept of one-switch utility functions and that of a stronger measure of risk aversion due to Ross (1981). We show however that specific assumptions about the behavior of the stronger measure of risk aversion also give rise to the linex utility function which belongs to the class of one-switch utility functions. In particular, this utility class is the only one that satisfies a stronger version of Kimball's (1993) standard risk aversion over all levels of wealth. We apply our results to consider $n$th-degree deteriorations in background risk and their effect on risk taking behavior.

*Key words and phrases:* Ross risk aversion, decreasing prudence, one-switch utility function.

*JEL code:* D81.

# 1 Introduction

In a stimulating paper, Bell (1988) argued that decision makers are likely to be characterized by a utility function satisfying the one-switch rule[1]. While four different families of utility functions satisfy such a rule, only one of them – the linear plus exponential, or linex – satisfies some desirable additional properties (including decreasing absolute risk aversion). Hence, this utility function seems to be relevant for many applications and its interest for the analysis of risky decisions was again illustrated in a recent paper by Sandvik and Thorlund-Petersen (2010).

Much earlier, Ross (1981) published an important paper devoted to an apparently different topic. Ross' paper extended a very basic result about risk averse behavior. Arrow (1965) and Pratt (1964) had previously shown that a more risk-averse decision maker is always willing to pay more for the elimination of a risk than a less risk-averse one. Ross (1981) extended this analysis to show that obtaining a similar result for risk reductions requires a stronger definition of increased risk aversion, which has become known in the literature simply as "increased Ross risk aversion."This was made clear and extended to higher orders of risk preferences by Jindapon and Neilson (2007) and Denuit and Eeckhoudt (2012a), who consider an opportunity, at some monetary cost, to shift from a less preferred distribution to one that dominates via $n$th-degree stochastic dominance. They show that an individual with higher $n$th-order Ross risk aversion will always shift more of the risk.

In the present paper, we show that Ross' (1981) notion of stronger risk aversion coupled with specific assumptions about its behavior reinforces the arguments in favor of a linex utility function besides those already suggested by Bell (1988) on very different grounds. We then apply this knowledge to extend a well-known result about behavior in the presence of a background risk. In particular, Eeckhoudt Gollier and Schlesinger (1996) provided conditions under which an increase in the riskiness of an independent background risk, via either first- or second-degree stochastic dominance, will lead to more risk-averse behavior. We show how linex utility not only satisfies their conditions, but it allows us to extend their analysis to an increase in the riskiness of the background risk by $n$th-degree stochastic dominance, for an arbitrary $n$. This helps us to better understand the background risk problem and its relation to Ross' definition of a stronger measure of absolute risk aversion as well as to its recent extension to higher degrees by Modica and Scarsini (2005) and by Denuit and Eeckhoudt (2010). This link is quite appropriate since in his paper, Ross discusses the value of partial risk reduction while a change in the background risk in the presence of a given foreground one can indeed be interpreted as a partial change in risk.

Our paper is organized as follows. In Section 2, we are more specific about each of the two contributions briefly described in this introduction. Section 3 indicates precisely how Ross' (1981) results and their extensions are related to the linex utility function. In Section 4, we apply our results to consider an increase in the riskiness of a background risk. We briefly conclude in Section 5. Proofs of the main results are gathered in an appendix.

[1]In this case, there exists at most one critical wealth level at which the decision maker switches from preferring one alternative to the other. See Section 2 for a definition and more information.

# 2 Bell's and Ross' contributions

Bell (1988) proposed "the one-switch rule as a reasonable property for utility functions to satisfy". The basic idea behind this rule can be summarized as follows. Consider two lotteries $X$ and $Y$ such that $E[X] > E[Y]$ while $X$ has the worst payoff. At low wealth levels, because the worst outcome is avoided by holding $Y$, this lottery is likely to be preferred by a risk averter. However, as wealth increases, the relative attractiveness of $Y$ decreases and there will exist a sufficiently high wealth level such that $X$ becomes preferred because of its higher mean. When at most only one such preference reversal occurs[2], the utility function satisfies the one-switch rule.

In his paper, Bell (1988) shows that only four families of utility functions satisfy the one-switch rule. The linear plus exponential (linex, in short) defined by

$$u(x) = lx - c\exp(-\gamma x) \tag{2.1}$$

where $x$ is wealth while $l$, $c$ and $\gamma$ are positive constants, is one of the four families. It turns out besides that it is the only relevant family if one adds to the one-switch rule some very reasonable requirements (e.g., the decision maker is risk averse at all wealth levels and his risk aversion is decreasing in wealth). Such utility functions have been studied by Bell and Fishburn (2001), among others.

Ross' (1981) paper deals with a very different topic. From Arrow (1965) and Pratt (1964) we know that if a decision maker becomes more risk averse, i.e. if his utility function is concavified[3], then he is willing to pay more for eliminating all risk in his wealth portfolio, i.e. the replacement of the risk by its expected value. Since such a result seems so intuitive, one expects it should also hold for a mean preserving contraction of a risk. Using simple examples, Ross (1981) shows that unfortunately the extension to marginal changes in risk (instead of its elimination) does not always work. He suggests a stronger definition of "more risk aversion" that leads to the desired result.

According to Arrow (1965) and Pratt (1964), $v$ is more risk averse than $u$ if the coefficient of absolute risk aversion $-\frac{v''(x)}{v'(x)}$ of $v$ exceeds the corresponding coefficient $-\frac{u''(x)}{u'(x)}$ for $u$ at all wealth levels $x$. Ross' (1981) stronger notion of risk aversion requires

$$-\frac{v''(x)}{v'(y)} \geq -\frac{u''(x)}{u'(y)} \text{ for all } x \text{ and } y. \tag{2.2}$$

Ross (1981) shows next that the stronger measure of more risk aversion is obtained when the more risk averse $v$ can be written as

$$v = \lambda u + \phi \tag{2.3}$$

where $\lambda$ is a positive constant and where $\phi$ is a non-increasing and concave function of wealth (i.e., $\phi' \leq 0$, $\phi'' \leq 0$).

Ross' (1981) concept also applies when, instead of comparing two individuals, one compares a given individual at two wealth levels. One then obtains the notion of stronger

[2]I.e. once $X$ becomes preferred, further increases in wealth no longer induce a return to $Y$.

[3]The utility function $u$ is concavified when the resulting function is $v(x) = k(u(x))$ with $k' > 0$ and $k'' < 0$.

decreasing absolute risk aversion (Ross DARA, in short). Whereas DARA implies that a decrease in wealth makes one behave in a more risk averse manner, Ross DARA implies that a stochastic decrease in wealth via first-order stochastic dominance makes one behave in a more risk averse manner, as shown by Eeckhoudt et al. (1996).

# 3 The link between these two contributions and the linex utility function

Bell (1988) had examined the conditions giving rise to the linex utility function without any reference to Ross' (1981) stronger measure. In this section, we show that the linex is a consequence of reasonable assumptions made about the behavior of Ross' stronger measure.

In the Arrow (1965) and Pratt (1964) environment as in Ross' (1981) one, it is straightforward to show that a utility function exhibits decreasing absolute risk aversion (DARA) whenever minus its marginal utility is more risk averse than itself. From (2.3), this implies that $u$ is Ross DARA if there exists a function $\phi$ such that $\phi' \leq 0$, $\phi'' \leq 0$, and a constant $\lambda > 0$ such that

$$-u' = \lambda u + \phi \tag{3.1}$$

Solving this differential equation reveals the functional form of every Ross DARA utility function, as formally stated in the next result.

**Property 3.1.** *A utility function $u$ satisfying* (3.1) *is of the form*

$$u(x) \;\;=\;\; \left(-\int \exp(\lambda x)\phi(x)dx\right)\exp(-\lambda x). \tag{3.2}$$

Taking $\phi(x) = -x$ in (3.2) gives

$$u(x) = -\frac{1}{\lambda^2} + \frac{x}{\lambda} - c\exp(-\lambda x) \tag{3.3}$$

for some positive constant $c$, where we recognize the functional form of the linex utility function defined in (2.1). Hence, with potentially many other utility functions, the linex is one of the utility functions that satisfy the Ross DARA condition[4].

In order to further explore the properties of the linex utility function in Ross' (1981) framework, we now consider an additional restriction that is often made in the expected utility framework. Following Kimball (1993), it is often assumed that preferences satisfy the property of "standard risk aversion," which in addition to DARA requires the property of decreasing absolute prudence (DAP in short). While this assumption has been extensively analyzed in the Arrow (1965) and Pratt (1964) setting, recent papers by Keenan, Rudow and Snow (2009) and Keenan and Snow (2012) suggest its use also in the context of Ross' (1981) stronger measure. Moreover, Ross DARA and Ross DAP are sufficient for any second-order stochastic dominance deterioration in a background risk to decrease risk-taking behavior, as shown by Eeckhoudt et al. (1996). We now show that the joint assumptions of Ross DARA and Ross DAP at all wealth levels lead exclusively to a linex utility function.

[4] Interestingly, when Ross (1981) illustrates his concept of stronger risk aversion he implicitly uses the linex utility as an example.

**Property 3.2.** *A risk averse utility function $u$ satisfying Ross DARA and Ross DAP at all wealth levels belongs to the linex class.*

The proof of this result is given in appendix. It is interesting to note that Ross prudence is not strictly decreasing here. Instead, it is constant everywhere. Our analysis shows that it is not possible to obtain both strict Ross DARA and strict Ross DAP at all wealth levels. Indeed, examples of both, as provided by Keenan and Snow (2012, Section 2.2) do not hold for all wealth levels, but only over a restricted range of wealth.

# 4 Changes in background risk and risk taking behavior

## 4.1 Stochastic dominance rules

Let us first recall the definition of the stochastic dominance rules used in the remainder of this paper. Consider a random variable $X$ valued in some interval $[a, b]$ of the real line, with distribution function $F_X$. Starting from $F_X^{[1]} = F_X$, define iteratively for $z \in [a, b]$

$$F_X^{[k+1]}(z) = \int_a^z F_X^{[k]}(t)dt$$

for $k = 1, 2, \ldots$. Then, $X$ is said to be smaller than $Y$ in $n$th-order stochastic dominance if $F_Y^{[n]}(z) \leq F_X^{[n]}(z)$ for all $z$, and if $F_Y^{[k]}(b) \leq F_X^{[k]}(b)$ for $k = 1, 2, \ldots, n-1$. This is denoted as $X \preceq_n Y$.

It is well known that $X \preceq_n Y$ is equivalent to $E[u(X)] \leq E[u(Y)]$ for all the utility functions $u$ with derivatives[5] $u^{(1)}, u^{(2)}, \ldots, u^{(n)}$ such that $(-1)^{k+1}u^{(k)} \geq 0$ for $k = 1, 2, \ldots, n$. Hence, $\preceq_n$ represents the common preferences of all the decision makers whose preferences satisfy risk apportionment of degrees 1 to $n$ in the terminology of Eeckhoudt and Schlesinger (2006). These decision makers prefer to disaggregate risk across equiprobable states of nature.

When the first $n-1$ moments of $X$ and $Y$ are equal, $n$th-order stochastic dominance coincides with the Ekern's (1980) concept of increase in $n$th-degree risk. As an example, $X$ is an increase in second degree risk over $Y$ if $X \preceq_2 Y$ and $E[X] = E[Y]$. This is what Rothshild and Stiglitz (1970) defined as a "mean-preserving increase in risk". Similarly, Menezes et al. (1980) described an increase in third-degree risk, which is also called an "increase in downside risk"corresponding to $X \preceq_3 Y$ with $E[X] = E[Y]$ and $E[X^2] = E[Y^2]$. More recently, Menezes and Wang (2005) defined an increase in outer risk, i.e. a 4th order increase in risk. All these contributions as well as their extensions to higher orders are synthesized by Eeckhoudt and Schlesinger (2006) who give both an intuitive interpretation of the signs of successive derivatives of a utility function and their link with $n$th degree increases in risk.

## 4.2 Shifts in $n$th stochastic dominance

Eeckhoudt et al. (1996) study conditions on preferences under which deteriorations in the distribution of the background risk entail more risk-averse behavior towards an endogenous

[5]In this paper, we use $u^{(j)}$ for the $j$th derivative of the function $u$. Low order derivatives are denoted as $u', u'', u''', u''''$ or $u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}$.

risk, such as investing in risky stocks as opposed to a risk-free bond. They establish that Ross DARA is necessary and sufficient for first-degree stochastic dominance deteriorations of a background risk to lead to more risk-averse behavior. Precisely, consider a decision maker with utility function $u$ and define the derived utility functions $v_1$ and $v_2$ as

$$v_i(w) = E[u(w + Y_i)], \quad i = 1, 2,$$

where $Y_1$ and $Y_2$ are two background risks affecting the agent's wealth $w$. Proposition 2 by Eeckhoudt, Gollier and Schlesinger (1996) indicates that any first-degree stochastic dominance deterioration in background risk raises local risk aversion at $w$ when $u$ is Ross DARA. Specifically, if (3.1) holds then

$$Y_1 \preceq_1 Y_2 \Rightarrow -\frac{v_1''(w)}{v_1'(w)} \geq -\frac{v_2''(w)}{v_2'(w)} \text{ for all } w. \tag{4.1}$$

These authors also consider second-degree stochastic dominance deteriorations of background risk. They show that decreasing Ross risk aversion together with a condition that is weaker than Ross DAP is both necessary and sufficient for second-degree stochastic dominance deteriorations of a background risk to lead to more risk-averse behavior. Precisely, if

(1) preferences exhibit Ross DARA and

(2) there exists $\lambda > 0$ such that the inequalities

$$-\frac{u''''(x)}{u'''(x)} \geq \lambda \geq -\frac{u''(x)}{u'(x)} \text{ hold for all } x \tag{4.2}$$

then

$$Y_1 \preceq_2 Y_2 \Rightarrow -\frac{v_1''(w)}{v_1'(w)} \geq -\frac{v_2''(w)}{v_2'(w)} \text{ for all } w. \tag{4.3}$$

The first derivative ratio in (4.2) is referred to as the measure of absolute temperance. Hence the second condition states that the minimum level of temperance exceeds the maximum level of risk aversion. To see how this fits in with Ross' analysis, recall that the condition (2.2) defining Ross DARA is equivalent to the existence of a constant $\lambda_1 > 0$ such that the inequalities

$$-\frac{u'''(x)}{u''(x)} \geq \lambda_1 \geq -\frac{u''(x)}{u'(x)} \tag{4.4}$$

hold for all $x$; see Theorem 4 in Ross (1981). Although not shown by Ross, it is easy to extend this result along lines used by Kimball (1990) to show that Ross DAP is equivalent to the existence of a constant $\lambda_2 > 0$ such that the inequalities

$$-\frac{u''''(x)}{u'''(x)} \geq \lambda_2 \geq -\frac{u'''(x)}{u''(x)} \tag{4.5}$$

hold for all $x$. It follows trivially from (4.4) and (4.5) that the properties of Ross DARA and Ross DAP jointly satisfy condition (4.2).

Although, not explicitly stated in the paper by Eeckhoudt et al. (1996), the implications (4.1)-(4.3) stil hold for higher order stochastic dominance under appropriate conditions imposed on the utility function $u$. Specifically, if there exists a constant $\lambda > 0$ such that the inequalities

$$-\frac{u^{(i+2)}(x)}{u^{(i+1)}(x)} \geq \lambda \geq -\frac{u''(x)}{u'(x)} \tag{4.6}$$

hold for all $x$ and for $i = 1, \ldots, n$ then any deterioration of a background risk via $n$th-degree stochastic dominance always leads to an increase in risk-averse behavior[6]. Note that, contrarily to the constants involved in (4.4)-(4.5), the constant $\lambda$ appearing in (4.6) does not depend on $i$ and is simply the supremum of the index of absolute risk aversion $-u''/u'$. Furthermore, as $u'' \leq 0$, the inequalities $-\frac{u^{(i+2)}}{u^{(i+1)}} \geq \lambda > 0$ for $i = 1, \ldots, n$ ensure that $u$ exhibits risk apportionment of orders 1 to $n + 2$. The proof follows easily from the one in the appendix of Eeckhoudt et al. (1996)[7].

Wang and Li (2011, Proposition 3.1) have shown that a similar result holds for an increase in $n$th-degree risk in the sense of Ekern (1980), which is weaker than stochastic dominance, if the inequality in (4.6) holds for $i = n$.

Let us now investigate the intermediate cases where $Y_1 \preceq_n Y_2$ and the first $j - 1 < n - 1$ moments of $Y_1$ and $Y_2$ coincide so that $Y_1$ is not an Ekern increase in risk with respect to $Y_2$. For instance, $j = 2$ corresponds to the mean-preserving stochastic dominance investigated in Denuit and Eeckhoudt (2012b). Since the empirical literature pays attention to the variance, skewness and kurtosis of the background (or foreground) risks, it is worth investigating how such changes in the background risk affect optimal decisions about the foreground one. The following result covers all the intermediate cases between Eeckhoudt, Gollier and Schlesinger (1996) and Wang and Li (2011), allowing for different moments of higher orders.

**Proposition 4.1.** *Let $n \geq 2$. Consider a decision maker with utility function $u$ such that condition (4.6) holds for $i = j, \ldots, n$. for some integer $j$ such that $1 \leq j \leq n$. Then,*

$$Y_1 \preceq_n Y_2 \text{ and } E[Y_1^k] = E[Y_2^k] \text{ for } k = 1, \ldots, j-1 \Rightarrow -\frac{v_1''(w)}{v_1'(w)} \geq -\frac{v_2''(w)}{v_2'(w)} \text{ for all } w.$$

The proof of this result is given in appendix. The limiting case $j = 0$ (in the sense that no moments of $Y_1$ and $Y_2$ coincide) corresponds to the result of Eeckhoudt, Gollier and Schlesinger (1996) suitably extended to arbitrary $n$, as explained above. If $j = n$ then $Y_1$ is an increase in $n$th-degree risk with respect to $Y_2$ and we are back to the setting of Wang and Li (2011).

## 4.3 The linex case

It is straightforward to confirm that the linex utility function, as described in (2.1), satisfies (4.6) for all $i$. Therefore, all the results derived earlier in this section also apply to the linex

[6]Following Jindapon and Neilson (2007), Wang and Li (2011) define decreasing $n$th-order Ross risk aversion via (4.6) for a specific $i = n$. They call Ross DAP "decreasing absolute risk prudence in the sense of Ross", abbreviated Ross DARP. This differs from decreasing 3rd-order Ross risk aversion. See Corollary 2.2 in Wang and Li (2011).

[7]Their proof is for $n$th-degree stochastic dominance, where they say to let $n = 1$ or $n = 2$; but there is no reason for this limitation as their proof is valid for any $n$, with very minor modifications.

case. The following illustration of a model of portfolio choice using linex utility gives us some insight into why this utility function might be so useful.

Consider the choice of allocating wealth between a risky and a risk-free asset. To this end, let $w$ denote the initial wealth of the individual who must decide on the amount of wealth $a$ to invest in the risky asset, with the remaining wealth $w - a$ invested in the risk-free asset. For notational ease, the return on the risk-free asset is assumed to be $r_f = 0$, whereas the risky return is denoted by the random variable $R$. We assume that $R \geq -1$ almost surely holds and that $E[R] > 0$. This last assumption guarantees that the optimal investment in the risky asset $a^\star$ will be strictly positive. There is a background risk $\mathcal{E}$, independent from $R$, such that $E[\mathcal{E}] \leq 0$.

The objective of the investor is to choose $a$ to maximize expected utility

$$U(a) = E[u(w + \mathcal{E} + aR)]. \tag{4.7}$$

It is easy to verify that $U$ is concave in $a$. Considering the linex utility function (2.1), we get

$$U(a) = lE[w + \mathcal{E} + aR] - cE[\exp(-\gamma(w + \mathcal{E} + aR))]$$

so that the first-order condition for a maximum under linex utility can be written as

$$\frac{dU}{da} = lE[R] + \gamma cE[\exp(-\gamma(w + \mathcal{E}))]E[\exp(-\gamma aR)R] = 0. \tag{4.8}$$

The parameter $l$ measures the relative importance of the linear (or risk neutral) part in the linex utility function, compared to the negative exponential (or CARA) part. It is therefore interesting to study the influence of this parameter on the optimal $a^\star$.

**Property 4.2.** *The solution $a^\star$ to* (4.8) *increases with $l$.*

This result simply follows from the fact that the partial derivative of (4.8) with respect to $l$ is $E[R] > 0$. Notice that in the special case of CARA utility $l = 0$, it follows from (4.8) that changes in $\mathcal{E}$ have no effect whatsoever on the optimal investment choice $a^\star$. If $l > 0$, we see from Property 4.2 that $a^\star$ will be higher than under CARA. Obviously, the linear (risk neutral) component of this utility function tempers risk aversion: when $l$ increases, the risk neutral component becomes relatively more important so that the demand for the risky asset increases.

It is reasonable to expect that the deterioration of the background risk has a tempering effect on the optimal investment in the risky asset when $l > 0$. The next result shows that this is indeed the case.

**Property 4.3.** *For $l > 0$, let $a_i^\star$ denote the investment choice for the background risk $\mathcal{E}_i$, $i = 1, 2$. If $\mathcal{E}_1 \preceq_n \mathcal{E}_2$ for some $n$ then $a_1^\star \leq a_2^\star$.*

The proof is given in appendix. It should be apparent from this proof that the existence of the constant term in the first-order condition $lE[R] > 0$, which derives from the linex utility, is what drives this result.

## 5 Conclusion

In the setting of Arrow (1965) and Pratt (1964), most currently used utility functions (logarithmic, power, mixture of exponentials) satisfy Kimball's (1993) property of standard risk aversion. That is, they jointly satisfy the conditions of DARA and DAP, in addition to being completely monotone. If we add an independent background risk - a type of market incompleteness - such preferences guarantee that a decision maker will take less risky decisions than she would in absence of the background risk.

When the Arrow (1965) and Pratt (1964) environment includes such a background risk, we need to move to Ross' (1981) stronger measure of risk aversion to make any further analyses. The same well-accepted conditions of DARA and DAP, if assumed to hold in this stronger Ross environment and assumed to hold at all wealth levels, yield a single class of utility functions: the linear plus exponential one. It thus appears that this (also completely monotone) utility function, justified a long time ago by Bell (1988) on very different grounds, has attractive features at the theoretical level, beyond those already known for the assessment of a decision maker's preferences.

The linex utility was also seen to be important in examining risk-taking behavior when the background risk deteriorates in the sense of becoming $n$th-degree riskier. In such cases the linex utility guarantees that the decision maker will take less risky decisions in the presence of the riskier background risk.

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# Appendix

## A Proof of Property 3.2

The utility function $u$ is Ross DAP if the representation

$$u'' = -\theta u' + \psi$$

holds for some function $\psi$ such that $\psi' \leq 0$, $\psi'' \leq 0$, and some constant $\theta > 0$. Combining this representation with (3.1) gives

$$u'' = -\theta u' + \psi = -\lambda u' - \phi' \tag{A.1}$$

which admits the following obvious solution:

$$\lambda = \theta \text{ and } -\phi' = \psi. \tag{A.2}$$

But this implies

$$\psi' = -\phi'' \geq 0$$

which contradicts the assumption $\psi' \leq 0$ unless it is satisfied via an equality. This can only be true if $\phi$ is linear, that is, if $u$ corresponds to a linear plus exponential utility function (3.3).

Notice that (A.2) may not be the unique solution to (A.1) if $\psi$ is a multiple of $u'$. However, this contradicts the assumption of risk aversion ($u'' \leq 0$), as shown next. Suppose there exists a constant $k$, with $u' = k\psi$ at every wealth level ($k$ might be positive or negative). From the properties of $\psi$, it would follow that $u''$ and $u'''$ have the same sign. But as Ross DARA implies regular DARA, we must have $u''' \geq 0$ which contradicts the existence of $k$.

## B Proof of Proposition 4.1

Clearly,

$$\begin{aligned}
-\frac{v_1''(w)}{v_1'(w)} \geq -\frac{v_2''(w)}{v_2'(w)} \quad &\Leftrightarrow \quad -v_1''(w) \geq -\frac{v_2''(w)}{v_2'(w)} v_1'(w) \\
&\Leftrightarrow \quad v_2''(w) - v_1''(w) \geq -\frac{v_2''(w)}{v_2'(w)} (v_1'(w) - v_2'(w)) \\
&\Leftrightarrow \quad \frac{E[u''(w+Y_2)] - E[u''(w+Y_1)]}{E[u'(w+Y_1)] - E[u'(w+Y_2)]} \geq -\frac{v_2''(w)}{v_2'(w)}
\end{aligned}$$

where the last equivalence comes from the fact that $-u'$ exhibits risk apportionment of orders 1 to $n+1$ so that $Y_1 \preceq_n Y_2 \Rightarrow E[u'(w+Y_1)] \geq E[u'(w+Y_2)]$. The following expansion formulas are easily obtained using repeated integration by parts:

$$\begin{aligned}
E[u''(w+Y_i)] &= \sum_{k=0}^{n-1} (-1)^k u^{(k+2)}(w+b) F_{Y_i}^{[k+1]}(b) + (-1)^n \int_a^b u^{(n+2)}(w+x) F_{Y_i}^{[n]}(x) dx \\
E[u'(w+Y_i)] &= \sum_{k=0}^{n-1} (-1)^k u^{(k+1)}(w+b) F_{Y_i}^{[k+1]}(b) + (-1)^n \int_a^b u^{(n+1)}(w+x) F_{Y_i}^{[n]}(x) dx.
\end{aligned}$$

Defining

$$\begin{aligned}
\alpha_1(w) &= (-1)^n \int_a^b u^{(n+2)}(w+x)\Big(F_{Y_2}^{[n]}(x) - F_{Y_1}^{[n]}(x)\Big)dx \geq 0 \\
\alpha_2(w) &= (-1)^n \int_a^b u^{(n+1)}(w+x)\Big(F_{Y_1}^{[n]}(x) - F_{Y_2}^{[n]}(x)\Big)dx \geq 0 \\
\beta_1(w) &= \sum_{k=j}^{n-1}(-1)^k u^{(k+2)}(w+b)\Big(F_{Y_2}^{[k+1]}(b) - F_{Y_1}^{[k+1]}(b)\Big) \geq 0 \\
\beta_2(w) &= \sum_{k=j}^{n-1}(-1)^k u^{(k+1)}(w+b)\Big(F_{Y_1}^{[k+1]}(b) - F_{Y_2}^{[k+1]}(b)\Big) \geq 0,
\end{aligned}$$

we have to show that

$$\frac{\alpha_1(w)+\beta_1(w)}{\alpha_2(w)+\beta_2(w)} \geq -\frac{v_2''(w)}{v_2'(w)}. \tag{B.1}$$

On the one hand, we have

$$\begin{aligned}
\frac{\alpha_1(w)}{\alpha_2(w)} &= \frac{\int_a^b u^{(n+2)}(w+x)\Big(F_{Y_2}^{[n]}(x) - F_{Y_1}^{[n]}(x)\Big)dx}{\int_a^b u^{(n+1)}(w+x)\Big(F_{Y_1}^{[n]}(x) - F_{Y_2}^{[n]}(x)\Big)dx} \\
&= \int_a^b -\frac{u^{(n+2)}(w+x)}{u^{(n+1)}(w+x)} \frac{u^{(n+1)}(w+x)\Big(F_{Y_1}^{[n]}(x) - F_{Y_2}^{[n]}(x)\Big)}{\int_a^b u^{(n+1)}(w+y)\Big(F_{Y_1}^{[n]}(y) - F_{Y_2}^{[y]}(x)\Big)dy}dx \\
&\geq \inf\left(-\frac{u^{(n+2)}(w+x)}{u^{(n+1)}(w+x)}\right) \\
&\geq \sup\left(-\frac{u''(w+x)}{u'(w+x)}\right) \text{ by (4.6) for } i=n \\
&\geq \int_a^b -\frac{u''(w+x)}{u'(w+x)} \frac{u'(w+x)dF_{Y_2}(x)}{\int_a^b u'(w+y)dF_{Y_2}(y)} \\
&= -\frac{v_2''(w)}{v_2'(w)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\beta_1(w)}{\beta_2(w)} &= \frac{\sum_{k=j}^{n-1}(-1)^k u^{(k+2)}(w+b)\Big(F_{Y_2}^{[k+1]}(b) - F_{Y_1}^{[k+1]}(b)\Big)}{\sum_{k=j}^{n-1}(-1)^k u^{(k+1)}(w+b)\Big(F_{Y_1}^{[k+1]}(b) - F_{Y_2}^{[k+1]}(b)\Big)} \\
&= \sum_{k=j}^{n-1} -\frac{u^{(k+2)}(w+b)}{u^{(k+1)}(w+b)} \frac{(-1)^k u^{(k+1)}(w+b)\Big(F_{Y_1}^{[k+1]}(b) - F_{Y_2}^{[k+1]}(b)\Big)}{\sum_{m=j}^{n-1}(-1)^m u^{(m+1)}(w+b)\Big(F_{Y_1}^{[m+1]}(b) - F_{Y_2}^{[m+1]}(b)\Big)} \\
&\geq \min_{k\in\{j,\ldots,n-1\}} \inf \left( -\frac{u^{(k+2)}(w+x)}{u^{(k+1)}(w+x)} \right) \\
&\geq \sup \left( -\frac{u''(w+x)}{u'(w+x)} \right) \text{ by (4.6) for } i = j, \ldots, n-1 \\
&\geq -\frac{v_2''(w)}{v_2'(w)}.
\end{aligned}$$

We have thus established that the inequalities

$$\frac{\alpha_1(w)}{\alpha_2(w)} \geq -\frac{v_2''(w)}{v_2'(w)} \text{ and } \frac{\beta_1(w)}{\beta_2(w)} \geq -\frac{v_2''(w)}{v_2'(w)}$$

both hold true, which in turn ensures that (B.1) is indeed valid. This ends the proof.

# C Proof of Property 4.3

Define the function $f$ as $f(\varepsilon) = \exp(-\gamma(w+\varepsilon))$ and note that $(-1)^i f^{(i)} \geq 0$ for all $i$. It thus follows (see, e.g. Jean, 1980) that $E[f(\mathcal{E}_2)] \leq E[f(\mathcal{E}_1)]$. Since $lE[R] > 0$, the term $E[\exp(-\gamma a R)R]$ in (4.8) must be negative at $a_2^\star$. If follows that investment in the risky asset will decrease for any deterioration in background risk via $n$th-degree stochastic dominance, i.e. when background risk $\mathcal{E}_1$ replaces background risk $\mathcal{E}_2$.