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Exposita notes

Arrow's theorem on the optimality of deductibles: A stochastic dominance approach*

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Summary. We provide a new proof for the optimality of deductible insurance that does not depend on the expected-utility hypothesis. Our model uses only first- and second-degree stochastic dominance arguments.

One of the most famous applications of the expected-utility hypothesis in insurance economics is due to Arrow (1971); "If an insurance company is willing to offer an insurance policy against loss desired by the buyer at a premium which depends only on the policy's actuarial value, then the policy chosen by a risk-averting buyer will take the form of 100 percent coverage above a deductible minimum." Raviv (1979) extended this result by considering Pareto-optimal insurance contracts. Arrow (1974), Buhlmann and Jewell (1979), Blazenko (1985), Gollier (1987a, b) and Gollier and Schlesinger (1994) have all added to Arrow's basic result.[1]

Zilcha and Chew (1990) and Karni (1992) show that Arrow's Theorem can be extended beyond the confines of expected-utility analysis.[2] In particular, if risk-averse preferences are defined via preference functionals consistent with first- and second-degree stochastic dominance, then Theorem 1 in Zilcha and Chew implies that Arrow's result holds in any model of choice under uncertainty. Existing proofs of Arrow's Theorem all have relied upon optimal control theory or the calculus of variation, and have shown that small perturbations from the optimal indemnity function decrease expected utility. Zilcha and Chew do not show Arrow's result directly, but rather rely on extent results in claiming "The proof of this result

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[1] A review of the surrounding literature on optimal insurance contracts in expected-utility models is provided by Gollier (1992).

[2] Karni extends Arrow's Theorem directly to preference functionals satisying Frechet differentiability, as introduced by Machina (1982) Karni's result, however, is a consequence of Theorem 1 in Zilcha and Chew, as Karni himself points out.

[Arrow's Theorem] relies heavily on the maximization of expected utility..." [Zilcha and Chew (1990, p. 130)]. Zilcha and Chew then invoke their Theorem, which shows the equivalence between efficient sets under risk-averse expected-utility maximization and stochastic-dominance preference, to show that Arrow's Theorem holds outside the expected-utility model.[3] In this paper, we show directly that a deductible insurance policy second-degree stochasticly dominates any other feasible insurance policy, without invoking the expected-utility hypothesis. Thus, the expected-utility version of Arrow's Theorem follows as a corollary to our proof given here.

There are two pedagogical advantages to our new proof of Arrow's result. First, since no knowledge of optimal control or calculus of variations is needed, our approach widens the accessibility of Arrow's result to a larger group of economists. Second, a direct stochastic-dominance analysis allows for a more complete understanding as to the unanimous preference of deductibles by all risk averters. As mentioned by Raviv (1979, p. 85), "A thorough understanding of [optimal policies] not only contributes to our understanding of insurance policies, but provides a foundation for the analysis of optimal contracts in more general situations."

Arrow's theorem

Let $(\Omega, \mathscr{F}, \mu)$ be a probability space and let $\mathrm{X}: \Omega \rightarrow \Re_+$ be a positive essentially-bounded function with essential supremum $W > 0$. We consider an individual with initial wealth W, which is subject to a random loss X. To protect against loss, the consumer may purchase an insurance contract by paying a premium P. The insurer is then bound to pay an indemnity $I(x)$ when loss x occurs, where $I(x)$ is some predetermined function.[4] The insurer is assumed to be risk-neutral. We examine the optimal form of $I(x)$ when P is fixed and when transaction costs only depend upon the actuarial value of the policy.

The individual's wealth following the purchase of insurance is given by the random variable Z,

$$Z \equiv W - P - X + I(\mathrm{X}) \tag{1}$$

Let G_I denote the distribution function (c.d.f.) for Z and let individual preferences over final wealth distributions be given by the functional $V(\cdot)$. The individual is assumed to be risk averse, which is defined here as $V(H) \geq V(G)$ if H dominates G in

[3] Zilcha and Chew essentially show that stochastic dominance is equivalent to a unanimous ranking by all real-valued functions $h(z)$ of the form $h(z) = \min(z, \theta)$ for some $\theta \in \Re$, which in turn is a basis for risk-averse utility functions. Thus, undominated wealth distributions from a given choice set of distribution functions are equivalent under the criteria of stochastic dominance and expected-utility risk aversion. One example of their result is the extension of Arrow's Theorem, which they provide for a finite probability space.

[4] We assume that the insurer meets its commitment. For an analysis of the effects of contract default, see Doherty and Schlesinger (1990).

the sense of second-degree stochastic dominance.[5] The objective here is to find the indemnity schedule $I(x)$ most preferred by the individual, subject to maintaining a fixed non-negative expected profit level for the insurer:

$$P - f(E[I(X)]) \geq k \geq 0, \tag{2}$$

where E above denotes the expectation operator. $E[I(X)]$ denotes the actuarial value of the policy with indemnity schedule $I(x)$. The function $f(EI)$ is the total expected cost for the insurer of the policy, including any transaction costs. We have

$$f(EI) \geq EI; \quad f'(EI) \geq 1. \tag{3}$$

We also restrict ourselves to non-negative indemnity payments that do not exceed the value of the loss, i.e.,[6]

$$0 \leq I(x) \leq x. \tag{4}$$

The problem now reduces to determining which function $I(x)$, chosen from all functions satisfying (2) and (4), is most preferred by the consumer. Let C denote the set of distribution functions for Z as given in (1), for all possible indemnity functions satisfying the inequality (4), i.e., $C = \{G_I | 0 \leq I(x) \leq x\}$. The objective is thus to

$$\underset{G_I \in C}{\text{maximize}}\ V(G_I) \text{ subject to (2)} \tag{5}$$

Define the random variable Y such that $Y := W - P - X$. Note that Y is obtained by taking the individual's original random wealth prospect and subtracting out the premium P. Thus, Y can be considered as the intermediate wealth position following the realization of a loss but prior to the payment of the insurance indemnity, $I(x)$.

Final wealth is given by adding in the insurance indemnity, $Z = Y + I(X)$. Constraint (2) together with specification (3) implies that $E(Z) \leq E(Y) + P - k$. The nonnegativity of $I(x)$ in constraint (4) implies that Z must be obtained from Y via a sequence of "rightward shifts" in the probability mass for final wealth; i.e. the distribution of Z must dominate Y in the sense of first-degree stochastic dominance (FSD). If this were not the case, then $I(x)$ would be negative for some values of x in contradiction to (4). Any increase in I which is compatible with the above constraints is desirable to consumers. It follows in a straightforward manner that the expected-profit constraint (2) is satisfied via an inequality if and only if $P - f(E[X]) > k$, that is, when constraint $I(x) \leq x$ is binding everywhere. Otherwise, there exist strategies – compatible with (2) and (4) – consisting of increasing cover-

[5] Second-degree stochastic dominance of H over G is defined as

$$\int_{-\infty}^{t} [G(z) - H(z)]dz \geq 0\ \forall t.$$

Note that second-degree in dominance is equivalent to a preference for fist-degree stochastic dominance $[G(t) \geq H(t) \forall t]$ and a preference against mean-preserving spreads [see Rothschild and Stiglitz (1970)]. If risk aversion is defined in the weaker sense of preference for receiving the mean of any wealth distribution with certainty, then deductible policies need no longer be optimal, as demonstrated by Safra and Zilcha (1988).

[6] See Gollier (1987a) for a discussion of optimal contracts when (4) does not hold

age for some levels of loss which would dominate the original indemnity schedule in the sense of the first-degree stochastic dominance. This case illustrates a premium that is so large that, even if full coverage ($I(x) = x$) is provided, the expected profit exceeds k. The optimal $I(x)$ function is trivially $I(x) = x$ in this case.

For the more interesting case where the above inequality does not hold, the optimal $I(x)$ function can be determined via second-degree stochastic dominance (SSD). To this end, let $H(\cdot)$ denote the distribution function for Y. Define $\underline{z}$ such that

$$\underline{z}H(\underline{z}) + \int_{\underline{z}}^{W-P} y dH(y) = E(Z), \tag{6}$$

and define

$$G(z) = \begin{cases} 0, & \text{if } z < \underline{z}; \\ H(z), & \text{if } z \geq \underline{z}. \end{cases} \tag{7}$$

Note that $\underline{z}$ is unique and that $G(z)$ is a well-defined distribution function for Z. In particular, note that Z as defined via $G(z)$ represents final wealth with a deductible insurance policy, where the deductible level is given by $d = W - P - \underline{z}$. We are now ready to state the following result of Arrow, which shows that the optimal insurance contract entails a straight deductible.

Theorem (Arrow): Given (1)–(4), $\exists d \geq 0$ *such that the optimal indemnity function* $I^*(x)$, *is given by*

$$I^*(X) = \max(0, x - d). \tag{8}$$

Proof: Fix P and $E[I(X)]$, and let the c.d.f. of Z be given by G as defined in (7). Since $I(x) \geq 0$ for all x, and $I(x) = 0$ for all x such that $z = W - P - x \leq \underline{z}$, for all $G_I \in C$, $G_I(z) \leq G(z)$ for all $z \geq \underline{z}$. To preserve the policy's actuarial value it must be the case that $G_I(z) \geq G(z)$ for some $z < \underline{z}$. Thus, any other cdf is a mean-preserving spread of G and consequently less preferred. (See Rothschild and Stiglitz (1970)).

Q.E.D.

This result is illustrated in Figure 1 for the case where $H(z)$ is continuous with associated density function $h(z)$. Then $G(z)$ is continous with density $g(z)$ identical to $h(z)$ for $z > \bar{z}$, but G has a mass point at $\bar{z}$. Any other G_I can only move probability

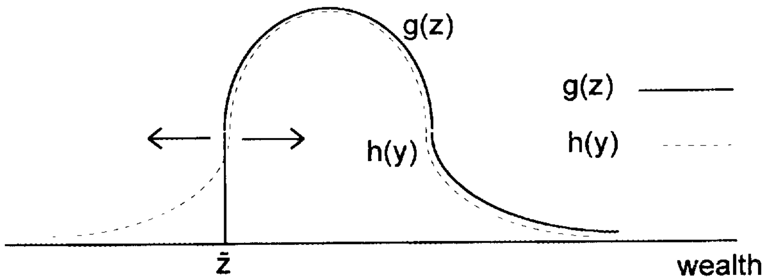

Figure 1.

mass to the right above $\bar{z}$. Therefore mass from $\bar{z}$ must shift left to preserve $E[I(x)]$, creating a mean-preserving spread.

If we allow P to be a decision variable in the above model, the consumer will select a level of coverage in which the cost of insurance balances the desirability of more coverage. The deductible result continues to hold once P is selected, since the deductible policy will dominate all other forms of coverage, according to the Theorem. The choice of P cannot be solved by a second-degree stochastic dominance argument and depends upon other attributes of consumer preferences, not only on the fact that consumers are risk-averse. If $f'(E(X)) > 1$, then Mossin (1968) has shown that less than full coverage will be chosen by an expected-utility maximizer, i.e. $d^* > 0$. If there are no transaction costs ($f(EI) = EI$), it is well-known that full insurance ($d^* = 0$) is optimal. Mossin's results also apply in nonexpected utility models, if risk aversion is of order 2, as defined by Segal and Spivak (1990). However, the results of Segal and Spivak imply that $d^* = 0$ also can be optimal when $f' > 1$, if risk aversion is of order 1.

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