Changes in Risk and the Demand for Saving

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Abstract

How does the risk inherent in future earnings or in interest rates affect the demand for saving? Whereas the theoretical literature mainly uses models with certainty as a point of reference, empirical work examines the effects of detectible differences in risk within the data. How these differences affect saving in theoretical models depends on the metric one uses for "risk." In the case of labor-income risk, we show how only second-degree changes in risk require prudence to induce increased saving. For first-order changes in risk, such as an increase in the probability of unemployment, prudence is not necessary. For higher order risk changes, prudence alone is no longer sufficient. For the case of an increase in interest rate risk, both a precautionary effect and a substitution effect need to be compared. In each case, we provide necessary and sufficient conditions on preferences for an N^{th}-degree change in risk to increase saving

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1 Introduction

The effect of risk on saving is a key element in understanding the intertemporal behavior of consumption. Early permanent-income-hypothesis models that lead to certainty equivalence have largely been replaced by models attempting to explain the observed positive correlation between income disturbances and consumption. To a large degree, these models assume that market incompleteness – the lack of a market for fully insuring particular types of risks – can explain some of these empirically observed phenomena.

Much of this literature has focused on labor-income risk at the household level. In addition to measurement errors, there is also the question as to which particular measure of risk is the most appropriate. As we show in this paper, the risk metric used is not innocuous. Moreover, whereas most empirical analyses deal with changes in risk, their theoretical counterparts all consider how models with uncertainty differ from those with certain income streams.[1]

[1] Good overviews of both the theoretical and empirical literature on this topic can be found in Kimball (1992), Browning and Lusardi (1996), Huggett and Ospina (2001) and Carroll and Kimball (2008).

A second and much smaller strand of literature has dealt with the effect of risky interest rates on saving. As opposed to the case where only future labor income is risky, a risky interest rate implies that "giving up a dollar's worth of certain present consumption does not result in a certain increase in future consumption." (Sandmo 1970, p. 353) Thus, a risky interest rate has different implications for its effect on saving than does risky labor income.

This paper considers very general types of changes in each of these two sources of risk: labor-income risk and interest-rate risk. In particular, we consider N^{th}-order stochastic-dominance changes in each risk, and we derive conditions on preferences that are both necessary and sufficient for these changes to lead to an increase in saving. In the case of labor-income risk, the increase in saving is due to a precautionary effect. In the case of risky interest income, both a precautionary effect as well as a type of substitution effect need to be weighed against each other to determine the overall effect on saving.

Establishing our results for general N^{th}-order changes in risk is not just a theoretical whim. Oftentimes establishing stochastic dominance of lower orders is not possible. Moreover, much of the extant empirical literature examines differences in risk with "risk" being measured using very different metrics. We explain in the next section how many of these metrics can be put into a stochastic-dominance framework. Our results are consistent with the premise that the higher moments of risky distributions (not just the first two moments) can be quite important. If such higher moments are affected by business cycles and other macroeconomic phenomena, this can lead to predictable saving behavior that has hitherto not been studied.

We proceed in the next section by defining both N^{th}-order stochastic dominance and an increase in N^{th}-degree risk, as defined by Ekern (1980). We show how several economic models of risk changes can be addressed by our set-up. We then turn to our general results for each source of risk. Since we derive conditions on preferences that are both necessary and sufficient to guarantee an increase in saving, our results have strong implications for empirical modeling. For instance, seemingly innocuous assumptions made about preferences might have stronger-than-intended implications.

In the case of labor-income risk, we show how the most commonly used utility functions will lead to an increase in precautionary savings for any N^{th}-order increase in risk. We then examine the same questions for a change in the random return on savings. Here the conditions on preferences that are needed for an increase in saving are more limiting, but they are satisfied for any N^{th}-order increase in risk under the common assumption of constant relative risk aversion (i.e. constant elasticity of substitution) whenever the coefficient of relative risk aversion exceeds one.

2 Types of Risk Changes

In this section, we define the types of risk changes that we study and we show how they apply within much of the extant literature on saving. We use a simple two-period setting in order to focus on the changes in risk. The individual is assumed to make a decision in the first period, when current income is known, about how much to consume and how much to save. In each model, either future labor income or the interest rate is stochastic. The results obtained in this paper can be embedded into more complex settings, such as multiple-period models or representative-agent general-equilibrium models, in well-known and straightforward manners.[2] This simplified setting allows us to better isolate the risk changes and their effects.

The types of risk changes we consider throughout the paper are based on the property of stochastic dominance. Let F and G denote two cumulative distribution functions of wealth, defined over a probability support contained within the open interval (a, b). Define $F_1 = F$ and $G_1 = G$. Now define $F_{n+1}(z) = \int_a^z F_n(t)dt$ for $n \geq 1$ and similarly define G_{n+1}. The distribution F dominates the distribution G via N^{th}-*order stochastic dominance* (NSD) if $F_N(z) \leq G_N(z)$ for all z, and if $F_n(b) \leq G_n(b)$ for $n = 1, ...N-1$. If the random wealth variables $\widetilde{y}$ and $\widetilde{x}$ have the

[2] A review of how such embeddings have been handled, as well as problems that might arise, can be found in Huggett and Ospina (2001).

distribution functions F and G respectively, we also will say that $\widetilde{y}$ dominates $\widetilde{x}$ via NSD.[3]

Following Ekern (1980), we characterize $\widetilde{x}$ as an *increase in* N^{th}*-degree risk* over $\widetilde{y}$ if $\widetilde{y}$ dominates $\widetilde{x}$ via N^{th}-order stochastic dominance and the first $N-1$ moments for the distributions of $\widetilde{y}_1$ and $\widetilde{x}_1$ coincide. As an example, $\widetilde{x}$ is an increase in second-degree risk over $\widetilde{y}$ if $\widetilde{y}$ dominates $\widetilde{x}$ via second-order stochastic dominance and both distributions have equal means. This is what Rothschild and Stiglitz (1970) define as a "mean-preserving increase in risk." Similarly, Menezes et al. (1980) describe a third-degree increase in risk, which they call an "increase in downside risk."

2.1 Labor Income Risk

We first consider risk in the distribution of future labor income. While several early models have used quadratic utility to show that certainty equivalence still holds in the face of labor-income risk, others were quick to show how some canonical higher-order properties of utility would lead to very different conclusions. In particular, a series of papers by Leland (1968), Sandmo (1970) and Dreze and Modigliani (1972) showed that when the utility function is separable, a positive third derivative generates a precautionary demand for saving. These papers all consider the change from a

[3] See, for example, Ingersoll (1987).

known non-stochastic level of future labor income to a stochastic future income of equal mean.

These authors seemed content to note that the common assumption of decreasing absolute risk aversion (DARA) was sufficient to generate this precautionary saving, although no intuition was provided as to why DARA might be of importance. Kimball (1990), who labels the property of a positive third derivative of utility as "prudence," was the first to more rigorously analyze how properties of preferences are linked to precautionary saving. Much of this analysis has been incorporated into the empirical literature on consumption and saving. However, the empirical literature has utilized various measures of risk, and for many of these risk measures prudence is neither necessary nor sufficient to generate a precautionary demand for saving.

Second-order risk increases

Most of the empirical literature considers various proxies of labor income, and then compares data with differing variances.[4] The underlying hypothesis is that a "riskier" labor-income stream will lead to higher precautionary saving. These empirical papers each assume a particular functional form for the distribution of labor income. If we abstract away from these distributional forms, we can infer that

[4] See Carroll and Kimball (2008) for a review of some of this literature.

these changes are focused on second-order increases in risk. The variance in these models serves as a proxy for "risk."[5]

As we show in this paper, such risk changes are a natural extension of the theoretical models mentioned above, which all consider certainty vs. uncertainty in future earnings. In particular, their theoretical underpinnings require prudence to explain how increased risk generates more saving. In addition, this literature has been extended to examine precautionary saving over the course of the business cycle. For example, Storesletten et al. (2004) show that idiosyncratic labor income risk is countercyclical, with a higher variance (leading to more saving) during downturns in the business cycle.

First-order risk increases

Of course, variance of labor income is not the only measure used within the empirical literature. Several authors point out difficulties with the data sets employed and with the proxies used in obtaining variance measures; and they use instead proxies for the probability of job loss.[6] However, an increase in the probability of becoming unemployed is not an increase in second-order risk. It is instead a first-order change in risk. An increase in the unemployment rate, by itself, would indicate a deterioration

[5]Generally, variance is not sufficient to guarantee a second-order increase in risk, as shown by Rothschild and Stiglitz (1970). However, if we restrict the universe to a particular functional form for the probability distributions, then variance may well be sufficient.

[6]For example, Carroll et al. (2003), Engen and Gruber (2001) and Lusardi (1988) all show that an increase in the probability of unemployment leads to an increase in precautionary saving.

in the distribution future labor income via first-order stochastic dominance.[7]

Again, such models generate predictions over the business cycle. For example, Parker and Preston (2005) show that precautionary saving is higher when the unemployment rate is expected to increase, but explain their result as being consistent with the notion that increased idiosyncratic risk is higher during a recession. If one considers the increased "risk" in future labor income to be a higher probability of unemployment, then we cannot use theoretical conclusions based upon prudence. Indeed, as will be shown, even a model with quadratic utility generates a precautionary demand for saving under first-degree risk increases.

In a similar vein, changes in unemployment benefits typically generate first-order changes in labor income risk. A decrease in benefit levels and/or tougher eligibility requirements leads to the same type of risk change as an increase in the probability of unemployment.[8]

Higher-order risk changes

Higher order risk changes also require different theoretical assumptions to generate precautionary saving. We are not aware of much empirical literature considering higher orders of risk changes. Guiso et al. (2001) do include an estimate of the third

[7] Interestingly, such a change need not increase the variance of future labor income. For example, as the probability of becoming unemployed approaches one, the variance would approach zero.

[8] See, for instance, Engen and Gruber (2001). As with much of the literature, the authors explain their precautionary demand via prudence, even though the type of risk change that they model is a first-order change.

moment of risky labor income, which adds a skewness dimension to the analysis, but they limit their analysis to a triangular distribution. This adds a third-order risk to the analysis. If the distribution of labor income is asymmetric, then how the skewness of the distribution changes over the course of a business cycle would seem to be an area ripe for empirical investigation.

In addition, changes in the tax structure might beget precautionary saving. For example, Davies and Hoy (2002) show how a change from a progressive income tax to a so-called "flat tax" might actually decrease the level of third-order risk within the distribution of post-tax labor income.[9]

Of course, fourth and higher orders of risk changes are also possible. For example, an empirical model might find that kurtosis varies in comparing distributions of labor income, even if the first three moments are mostly identical. Would such a finding have an implication for precautionary saving?

2.2 Interest Rate Risk

The second source of risk that we consider is the interest rate on saving. To the best of our knowledge, such models were first formalized by Phelps (1962) and Levhari and Srinivasan (1969). Characterizations of how saving in the presence of interest-

[9] In actuality, Davies and Hoy (2002) consider a redistribution of wealth within the population. However, we can reinterpret their results in terms of earnings probabilities.

rate risk compares with the no-risk case were first developed by Hahn (1970) and by Sandmo (1970).[10] Such cases require one to balance a substitution effect (reducing saving if the return on savings becomes riskier and, hence, less attractive) against a precautionary effect. If the precautionary motive is strong enough, in a way that we make precise in this paper, then saving will increase.

An extension to mean-preserving increases in risk was developed by Rothschild and Stiglitz (1971). But what of other types of stochastic changes in the interest rate? What will be the effect on saving? To our knowledge, no one has examined the effects of other orders of risk changes.

Most obviously, first-order risk changes would seem fairly common in real-world markets. For example news events might indicate that short-term interest rates are expected to be stochastically lower: the conditional distribution of interest rates would deteriorate in the sense of first-order stochastic dominance. But what effect, if any, would this new distribution have on precautionary saving?

Higher order risk changes also would appear to be relevant. Empirical models of short-term nominal interest rates have observed that the third and fourth moments of the distribution are important. For example, Engle (1982) explains observed

[10] It was interesting to us that the paper by Hahn (1970) has received relatively little attention in the literature. This may be partly due to the informal structure in Hahn's paper. He uses variance as his risk measure, but uses a heuristic graphical argument with a Bernoulli distribution for interest rates.

leptokurtosis in the distribution of short-term nominal rates via heteroskedasticity. Gray (1996) obtains a similar result using a generalized regime switching model. Dutta and Babbel (2005) compare alternative functional forms of the distribution function and find that both skewness and kurtosis are significantly higher than with the standard lognormal distribution.[11]

Since changes in skewness and kurtosis are respectively third- and fourth-order changes in risk, we need to ask how these measures change with respect to macroeconomic policy and/or over the business cycle. And, how do these changes affect the demand for precautionary saving?

3 Labor Income Risk and Precautionary Saving

We consider a consumer with a two-period planning horizon. As a base case, consider a consumer with a certain income stream of y_0 at date $t = 0$ and y_1 at date $t = 1$. The consumer must decide how much to save at date $t = 0$. Any income that is not saved is consumed at that date. There is a fixed rate of interest r for both saving and for borrowing. Savings are allowed to be negative (i.e. we allow for borrowing), so long as the amount borrowed can be repaid from the future income y_1.

[11] The lognormal is "standard" if we model interest rates via diffusion processes that are based on geometric Brownian motion.

The consumer is assumed to have preferences that are intertemporally separable with a preference for smoothing consumption over time. To this end, the consumer chooses a level of saving so as to maximize her lifetime utility of consumption:

$$\max_s U(s) \equiv u(y_0 - s) + \frac{1}{1+\delta} u(y_1 + s(1+r)), \tag{1}$$

where the utility function u is assumed to be strictly increasing and strictly concave and where δ represents the consumer's personal rate of discount for delaying the utility of future consumption. We also assume throughout this paper that u is continuously differentiable

The first-order condition for (1) is

$$U'(s) = -u'(y_0 - s) + \frac{1+r}{1+\delta} u'(y_1 + s(1+r)) = 0, \tag{2}$$

which we assume to hold at some interior value s^*, $-y_1 < s^*(1+r) < y_0(1+r)$. It also follows trivially from (1) that $U(s)$ is strictly concave in s, whenever the utility function u is strictly concave in consumption. As a result, the second-order condition for a maximum holds and s^* is unique. For example, if $r = \delta$, then as is well known, saving is used as a device to perfectly eliminate fluctuations in consumption over time.

Now assume that labor income at date $t = 1$ is risky, say $\widetilde{y}_1$, where $E\widetilde{y}_1 = y_1$. In this case, it is not possible to perfectly smooth consumption over time. Indeed, consumption at date $t = 1$ depends on the realized value of future labor income $\widetilde{y}_1$. Thus, the objective of the consumer becomes

$$\max_s U(s) \equiv u(y_0 - s) + \frac{1}{1+\delta} Eu(\widetilde{y}_1 + s(1+r)). \tag{3}$$

It follows in a straightforward manner from (3) that the optimal level of saving will necessarily increase whenever $Eu'(\widetilde{y}_1 + s^*(1+r)) \geq u'(y_1 + s^*(1+r))$, which can be guaranteed for any arbitrary values of y_1 and r, and any arbitrary mean-y_1 random variable if and only if u' is a convex function of consumption, i.e. if and only if $u''' \geq 0$.[12] This increased level of saving due to the labor income risk is precisely the precautionary part of total saving.

Some insight into this result is provided by Menezes and Hansen (1971), who related $u''' > 0$ to the utility premium of Friedman and Savage (1948). The utility premium simply measures the difference in utility between $u(w)$ and $Eu(w + \widetilde{\varepsilon})$ for any random wealth variable $\widetilde{\varepsilon}$. If $E\widetilde{\varepsilon} \leq 0$, this utility premium simply measures

[12] We will consider an "increase" in savings to be in the weak sense that savings does not fall. All of our results below extend to strict inequalities by well known methods. However, this leads to more complicated mathematical conditions, with no real gain in economic insight. Hence, we consider only increases in savings in this weak sense throughout the paper.

the loss of utility from adding the risk $\widetilde{\varepsilon}$ to one's wealth. As such, we can refer to this utility premium as an intrapersonal measure of "pain" for adding the risk $\widetilde{\varepsilon}$. [13] Prudence implies that the "pain" of the risk $\widetilde{\varepsilon}$ is lower at higher wealth levels. In the case of labor-income risk, the individual can reduce the "pain" of the riskiness of $\widetilde{y}_1$ by shifting a bit more wealth from date $t = 0$ to the date $t = 1$.

We now extend the analysis to more general risk changes based on various orders of stochastic dominance. Denote the level of saving that maximizes (3) as s_y. Thus, s_y is the solution to the first-order condition

$$U'(s) = -u'(y_0 - s) + \frac{1+r}{1+\delta} Eu'(\widetilde{y}_1 + s(1+r)) = 0. \tag{4}$$

Consider a change in random wealth at date $t = 1$ from $\widetilde{y}_1$ to $\widetilde{x}_1$, where $\widetilde{y}_1$ dominates $\widetilde{x}_1$ via N^{th}-order stochastic dominance. As a trivial example, suppose $\widetilde{y}_1 = y_1$ and $\widetilde{x}_1 = x_1$ are both constants, with $y_1 > x_1$. It follows trivially from (2) that saving will increase whenever u' is decreasing, i.e. $u'' \leq 0$. But this also turns out to be a necessary and sufficient condition for any stochastic change in labor income for which $\widetilde{y}_1$ dominates $\widetilde{x}_1$ by first-order stochastic dominance (FSD).

Using (4), we see that saving will increase whenever $Eu'(\widetilde{y}_1 + s_y(1+r)) \leq Eu'(\widetilde{x}_1 +$

[13] Eeckhoudt and Schlesinger (2006) introduce the utility premium as a measure of pain, and they use it to characterize higher-order preference effects in much the same way as this characterization of prudence.

$s_y(1+r))$ for any arbitrary value of r. Under first-order stochastic dominance, this inequality holds for any arbitrary $\widetilde{y}_1$ and $\widetilde{x}_1$ exhibiting FSD if and only if u' is a decreasing function. In other words, a consumer always saves more when future labor income becomes riskier in the sense of FSD if and only if the consumer is risk-averse.

But what about higher order stochastic changes in future labor income? Proposition 1 below extends the analysis to any arbitrary degree of stochastic dominance N, where $\widetilde{y}_1$ dominates $\widetilde{x}_1$ via N^{th}-order stochastic dominance (NSD). The proof of the Proposition hinges on the following well known equivalence result:

NSD Equivalence[14]

The following two statements are equivalent

(i) $\widetilde{y}_1$ *dominates* $\widetilde{x}_1$ *via NSD*

(ii) $Ef(\widetilde{y}_1) \leq Ef(\widetilde{x}_1)$ *for any arbitrary function* f *such that* $\operatorname{sgn}(d^n f(t)/dt^n) = (-1)^n$, *for all* $n = 1, 2, ..., N$.

We are now able to state our main result. For notational convenience, we denote $f^{(n)}(t) \equiv d^n f(t)/dt^n$.

Proposition 1 *Given a two-period consumption and saving problem as specified in (3), with risky future labor income* $\widetilde{y}_1$ *or* $\widetilde{x}_1$*, the following two statements are equiv-*

[14] See, for example, Ingersoll (1987).

alent:

(i) The optimal level of saving under $\widetilde{x}_1$ *is always as least as high as under* $\widetilde{y}_1$*, for every utility function* u *such that* $sgn[u^{(n)}(t)] = (-1)^{n+1}$*, for* $n = 1, 2, ..., N+1$

(ii) $\widetilde{y}_1$ *dominates* $\widetilde{x}_1$ *via NSD.*

Proof. We require $u' > 0$ and $u'' < 0$ for (3) to be well defined. The result here follows by simply defining $f(t) \equiv u'(t + s_y(1+r))$ in the NSD equivalence statements, where s_y is the solution to (4). ■

Proposition 1 can be easily extended to the following Corollary, which follows in a straightforward manner by using Ekern's definition of an increase in N^{th}-degree risk.

Corollary 1: *Given a two-period consumption and saving problem as specified in (3), with risky future labor income* $\widetilde{y}_1$ *or* $\widetilde{x}_1$*, the following two statements are equivalent:*

(i) The optimal level of saving under $\widetilde{x}_1$ *is as least as high as under* $\widetilde{y}_1$*, for every utility function* u *such that* $\text{sgn}[u^{(N+1)}(t)] = (-1)^N$

(ii) $\widetilde{x}_1$ *is an* N^{th}*-degree increase in risk over* $\widetilde{y}_1$*.*

Note that in the "standard" model of precautionary saving, in which we go from a fixed future labor income to a random one, we can think of $\widetilde{y}_1 = y_1$ and $E\widetilde{x}_1 = y_1$. This is just a special case of Proposition 1 with $N = 2$. Since the precautionary part of saving is zero when the future labor income is not risky, we obtain that $u''' \geq 0$ is both necessary and sufficient for the existence of precautionary saving.

Returning to the examples of risk used in section 2, we see that prudence is both necessary and sufficient to guarantee that any second-order increase in labor-income risk leads to higher precautionary saving. On the other hand, a higher probability of unemployment or a reduction in unemployment benefits will lead to higher precautionary saving under risk aversion alone, since these both represent first-order risk changes. Prudence is not necessary in this case. Even the oft maligned quadratic utility function, which is not prudent, is sufficient to generate precautionary saving in this case.

For cases with third-order risk changes, prudence alone is not sufficient to generate an increase in precautionary saving. This would require the condition of "temperance," $u^{(4)}(t) \leq 0$.[15] Thus, for example, if a switch to a flat tax on income generated a reduction in downside risk, as described in section 2, it would lead to less precautionary saving under temperance.

[15] Although not much has been written about higher-order effects, there are a few papers. The term "temperance" follows from Kimball (1992). The condition $u^{(5)}(t) \geq 0$ is referred to as "edginess" (Lajeri-Chaherli, 2004). Although these authors find such conditions as necessary for various types of behavior, they do not offer any settings under which these conditions are sufficient for anything.

4 Risky Interest Rates

We now assume that the interest paid on savings is itself random, rather than fixed. Even so-called "risk-free" bonds are typically only risk-free with respect to any default risk. There still might be some risk that the market rates will change, or that real purchasing power cannot be guaranteed due to unexpected inflation. Rothschild and Stiglitz (1971) considered a savings model with income only at date $t = 0$. At this date, the consumer decides how much to consume and how much to save, for consumption at date $t = 1$. Any amount saved earns a rate of interest $\widetilde{r}$, where we assume $\widetilde{r} > -1$. The consumer's objective can thus be written as

$$\max_s U(s) \equiv u(y_0 - s) + \frac{1}{1+\delta} Eu(s\widetilde{R}), \tag{5}$$

where, for ease of notation, $\widetilde{R}$ denotes the gross rate of interest, $\widetilde{R} = (1 + \widetilde{r})$.

The first-order condition for (5) is

$$U'(s) \equiv -u'(y_0 - s) + \frac{1}{1+\delta} E[u'(s\widetilde{R})\widetilde{R}] = 0. \tag{6}$$

It is again straightforward to show that $U(s)$ is strictly concave, so that second-order conditions for a maximum easily hold and any solution to (6) is unique. We assume that the optimal level of saving s^* is strictly positive, with $s^* < y_0$.[16]

Rothschild and Stiglitz (1971) consider a change in the distribution of $\widetilde{R}$ to one that is a mean-preserving increase in risk, as defined by their earlier paper Rothschild and Stiglitz (1970). We examine a more general stochastic change in interest from say $\widetilde{R}_a$ to $\widetilde{R}_b$. We first consider the case where $\widetilde{R}_a$ dominates $\widetilde{R}_b$ by NSD.

Proposition 2 *Let s_i denote the optimal level of saving for (5) when $\widetilde{R} = \widetilde{R}_i$, for $i = a, b$. The following two statements are equivalent:*

(i) The optimal level of saving under $\widetilde{R}_b$ is as least as high as under $\widetilde{R}_a$, for every utility function u such that $[-tu^{(n+1)}(t)]/u^{(n)} \geq n$ for all $n = 1, 2, ..., N$

(ii) $\widetilde{R}_a$ dominates $\widetilde{R}_b$ via NSD.

Proof. From (6) and the concavity of $U(s)$, we know that $s_b \geq s_a$ whenever $E[u'(s_a\widetilde{R}_b)\widetilde{R}_b] \geq E[u'(s_a\widetilde{R}_a)\widetilde{R}_a]$. From NSD equivalence, this will hold for all y_0 whenever the function $h(R) \equiv Ru'(s_aR)$ satisfies the property that $\mathrm{sgn}(h^{(n)}(R)) = (-1)^n$ for all $n = 1, 2, ..., N$. We proceed by induction. For $N = 1$, straightforward calculation shows that $h'(R) = s_aRu''(s_aR) + u'(s_aR)$. Thus $h'(R) \leq 0$ holds for all R if and only if relative risk aversion is greater than one: $-tu''(t)/u'(t) \geq 1$, $\forall t > 0$.

[16] Since we assume $\widetilde{r} > -1$, the assumption of $s^* > 0$ avoids any issues associated with bankruptcy. Taken together with the assumption that $s^* < y_0$, we are simply assuming that the consumer must have positive consumption in each period.

For any $n > 1$, it follows from standard induction arguments that

$$h^{(n)}(R) = (s_a)^n R u^{(n+1)}(s_a R) + n(s_a)^{n-1} u^{(n)}(s_a R). \tag{7}$$

Since by assumption s_a is strictly positive, it follows that $h^{(n)}(R) \leq [\geq] 0$ for all R if and only if $-tu^{(n+1)}(t)/u^{(n)}(t) \geq [\leq]\ n$, $\forall t > 0$, so long as $u^{(n)}(t) \neq 0$. Applying NSD equivalence, it follows that $\widetilde{R}_a$ dominates $\widetilde{R}_b$ via NSD is equivalent to $E[u'(s_a\widetilde{R}_b)\widetilde{R}_b] \geq E[u'(s_a\widetilde{R}_a)\widetilde{R}_a]$ for all utility u such that $-tu^{(n+1)}(t)/u^{(n)}(t) \geq\ n$, $\forall t > 0$, $\forall n = 1, 2, ..., N$. The result then follows immediately.

If $u^{(n+1)}(t) = 0$ for all $t > 0$, then $h^{(n)}(R)$ is a constant, possibly identical to zero. Consequently, any n^{th}-degree increase risk for interest rates will have no effect on the level of optimal level of saving. ■

In a manner similar to Corollary 1, we can induce from Proposition 2 the following result.

Corollary 2: *Let s_i denote the optimal level of saving for (5) when $\widetilde{R} = \widetilde{R}_i$, for $i = a, b$. The following two statements are equivalent:*

(i) The optimal level of saving under $\widetilde{R}_b$ is as least as high as under $\widetilde{R}_a$, for every utility function u such that $[-tu^{(N+1)}(t)]/u^{(N)}(t) \geq N$, so long as $u^{(N)}(t) \neq 0\ \ \forall t > 0$. For $u^{(N)}(t) = 0$, the saving levels will be identical.

(ii) $\widetilde{R}_b$ *is an* N^{th}*-degree increase in risk over* $\widetilde{R}_a$*.*

The result of Rothschild and Stiglitz (1971) is a special case of Corollary 2, with $N = 2$: If $\widetilde{R}_b$ is a mean-preserving increase in risk over $\widetilde{R}_a$, then $s_b \geq s_a$ whenever relative prudence exceeds two. As another example, suppose that economic forecasters predict that economic performance will be less than previously expected. In this case, we might expect that $\widetilde{R}_b$ is first-order increase in risk over $\widetilde{R}_a$. Hence, saving will increase ($s_b \geq s_a$) whenever relative risk aversion exceeds one.

It is noteworthy to compare the results from the two different set-ups. For the sake of concreteness, consider an increase in third-order risk, $N = 3$. In the model where the rate of interest was fixed, but future labor income was risky, we required only that the consumer be temperate, in order for a third-order increase in risk in the distribution of future labor income to increase precautionary saving.

Consider the model in which only the interest rate is risky. This case would correspond to changes in the skewness of short-term interest rates. Here there are, in a sense, two effects. This can be seen by examining (7), which can be written as $h'''(R) = (s_a)^3 R u''''(s_a R) + n(s_a)^2 u'''(s_a R)$. If u were cubic with $u''' > 0$ but $u'''' = 0$, we would have $h'''(R) > 0$ due to prudence. In this case, an individual who is prudent would save less. The reason for this is that savings itself would increase the level of third-order riskiness in second-period consumption. Thus, the individual would opt for more consumption at date $t = 0$ (i.e. less saving). This can viewed as a type of substitution effect, due to the deterioration (increased third-order riskiness) of the saving instrument. On the other hand, if the consumer is temperate, $u'''' < 0$, then the extra third-order risk for consumption at date $t = 1$ will induce precautionary saving. We can refer to this as a precautionary effect. For the net effect to be an increase in saving, the precautionary effect must dominate, which will be the case if and only if relative temperance exceeds three.[17]

In the case with $N = 3$, we thus require more than just relative prudence in excess of two. On the other hand, consider the simple case of a first-order change in interest-rate risk, such as the stochastically lower interest rates mentioned in section 2. Here, relative prudence in excess of two is not even necessary, so long as relative risk aversion exceeds one. If we have logarithmic utility (with relative risk aversion equal to one), then first-order risk changes will have no effect on saving: the positive precautionary effect will be precisely offset by the negative substitution effect.

It is interesting to note as example a few particular utility functions, that are quite common within the macroeconomics literature.

[17] For $N = 4$, we would require $-tu^{(5)}(t)/u^{(4)}(t) \geq 5$, which using nomenclature from Lajeri-Chaherli (2004) we can describe as the measure of "relative edginess" exceeding five. For $n \geq 6$, we are unaware of any literature describing or naming the measure $-tu^{(n)}(t)/u^{(n-1)}(t)$. However, the general condition under expected utility that $\text{sgn}[u^{(n)}] = (-1)^{n+1}$ is discussed by Caballe and Pomansky (1995), who label $-u^{(n)}(t)/u^{(n-1)}(t)$ as a measure of "n^{th}-degree risk aversion." We thus might wish to label $-tu^{(n)}(t)/u^{(n-1)}(t)$ as a measure of "relative n^{th}-degree risk aversion."

Example 1: Consider the case of quadratic utility, with $u(t) = t - kt^2$, where $k > 0$ and we restrict $t < (2k)^{-1}$. If we further restrict t such that $(4k)^{-1} < t < (2k)^{-1}$, then relative risk aversion exceeds one, so that any first-degree increase in risk from $\widetilde{R}_b$ to $\widetilde{R}_a$ will increase saving. But an increase in N^{th}-degree risk for any N other than $N = 1$, will lead to no change in saving, since relative prudence is zero.

Example 2: For utility that belongs to the class of functions exhibiting constant absolute risk aversion (CARA), with $-tu''(t)/u'(t) = \theta t \quad \forall t > 0$, where $\theta > 0$, it follows that $[-tu^{(N+1)}(t)]/u^{(N)}(t) = \theta t$ as well. Thus, an N^{th}-degree increase in interest-rate risk will lead to higher saving only if wealth is sufficiently high; in particular $t \geq N/\theta$.

Example 3: For utility that belongs to the often-used class of functions exhibiting constant relative risk aversion (CRRA), with $-tu''(t)/u'(t) = \gamma \quad \forall t > 0$, where $\gamma > 0$, it follows that $[-tu^{(N+1)}(t)]/u^{(N)}(t) = (\gamma - 1) + N$, which of course exceeds N whenever $\gamma \geq 1$.[18] Thus, for CRRA utility, relative risk aversion larger than one is both necessary and sufficient for any N^{th}-degree increase in risk to increase the level of saving.

[18] CRRA utility takes the form $u(t) = \ln t$ or $u(t) = \frac{1}{1-\gamma} t^{1-\gamma}$ for $\gamma \neq 1$.

5 Concluding Remarks

We examined two models of saving with a risky future. When future labor income is risky, a stochastic increase in N^{th}-degree risk leads to more saving if and only if $\text{sgn}[u^{(N+1)}] = (-1)^N$. When there is an increase in N^{th}-degree risk in the return on savings, this condition is no longer sufficient, and we require that the measure of relative $(N+1)^{st}$-degree risk aversion exceed N, i.e., $[-tu^{(N+1)}(t)]/u^{(N)}(t) \geq N$. This condition guarantees a precautionary effect that is strong enough to increase the level of saving.

Prudence has often been considered as synonymous with a precautionary demand for saving in the face of increased labor-income risk. However, this turns out to hold only for second-degree risk changes. For example, a higher probability of unemployment or other first-order risk changes would lead to an increased precautionary demand under the weaker condition of risk aversion.

Risk changes of degree higher than two have been examined empirically for interest-rate risk but not examined very much for labor income risk. This might be due, in part, to not having any theoretical predictions about what such risk changes would entail. Hopefully this paper can contribute to further research in this direction.

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