Mossin's Theorem for Upper-Limit Insurance Policies*

Harris Schlesinger
Department of Finance, University of Alabama, USA
Center of Finance & Econometrics, University of Konstanz, Germany
E-mail: hschlesi@cba.ua.edu

March 10, 2005

Abstract

Although Mossin's Theorem ("full insurance with a fair premium and less-than-full coverage with a proportional premium loading") is well known for the classes of coinsurance contracts and for deductible- insurance contracts, it has not been proven for the class of upper-limit insurance contracts. This paper provides a proof for this case.

Keywords: Insurance, Mossin's Theorem, Upper Limits, Deductibles
JEL Classification: D81

*The author wishes to thank the students in his Insurance Management class at the University of Konstanz for posing the questions that led to this paper. He also thanks two anonymous referee's for their comments.

Mossin (1968) showed that full insurance would be purchased by a risk-averse individual when the premium is actuarially fair, i.e. when the premium is equal to the expected indemnity payment by the insurer. This is hardly surprising, since it merely states that a risk averter will prefer a final wealth level with no risk, to any other wealth distribution with the same expected level of final wealth, but with a positive variance. If the premium includes a proportional premium loading factor $\lambda > 0$, then Mossin showed that a policy paying less-than-full coverage would be optimal.[1] Although Mossin considered only the class of proportional coinsurance, his result extends easily to deductible insurance. For a summary of these cases see Schlesinger (2000). Indeed, Mossin's Theorem is so intuitive that it would seem to trivially apply to any constrained class of insurance contracts, such as to upper-limit policies. But as it is sometimes said, "the devil is in the detail," and there are a couple of interesting "twists" along the way.

In this paper, I provide a proof of Mossin's Theorem for the restricted class of insurance policies with an upper limit on coverage. Such a contract provides full coverage for a loss up to some pre-specified level θ. Following a loss of size x, the indemnity for such a contract is specified as $I(x) = \min(x, \theta)$. For example, such policies are common for liability contracts.

1 Basic Problem

To allow for an analytic solution, I restrict the analysis to the case where the distribution of losses is continuous, with density function $f(x)$ and $F(x)$ denotes the cumulative distribution. The individual has a terminal wealth denoted by the random variable $\widetilde{y}$ and preferences under risk are characterized by the utility function u, where u is increasing and strictly concave due to risk aversion.

We can write the expected utility of terminal wealth as follows:

$$Eu(\widetilde{y}) \equiv \int_0^\theta u(W - P(\theta))f(x)dx + \int_\theta^L u(W - P(\theta) - x + \theta)f(x)dx \qquad (1)$$

where

$$P(\theta) = (1+\lambda)\left[\int_0^\theta xf(x)dx + [1 - F(\theta)]\theta\right]. \qquad (2)$$

Here W denotes initial wealth, L denotes the upper limit of the support of $\widetilde{x}$ and P denotes the insurance premium. To avoid bankruptcy complications, I assume that $0 \leq x \leq L \leq W$. The consumer's problem is to choose an upper limit θ so as to maximize $Eu(\widetilde{y})$ in (1).

The problem is that for $\theta > L$ the premium as defined in (2) will not go up, regardless of the premium loading λ. This is because there is no extra cost

[1] Note that Mossin considers only a proportional premium loading when deriving his result. If insurance premia equal the actuarial cost plus a constant loading, then it is trivial to show (since the marginal price of insurance coverage is always "fair") that full coverage is optimal whenever the fixed loading is smaller than the individual's Arrow-Pratt risk premium.

to the insurer – losses never exceed L. For example, suppose that you own an asset with a maximum worth of $x = \$10,000$. A policy with an upper limit of $\theta = \$11,000$ will cost no more than one with $\theta = \$10,000$. Since losses never exceed $\$10,000$, the actuarial formula in (2) requires no additional premium.

As a result, expected utility as a function of θ might look as drawn in Figure 1 (drawn for $\lambda > 0$). Solving for $dEu/d\theta = 0$ will not necessarily yield a local maximum. Indeed, in a neighborhood of $\theta = L$ expected utility cannot be concave in θ.

——— INSERT FIGURE 1 ABOUT HERE ———

2 Mossin's Theorem

Let $\overline{\theta} < L$. For a zero loading, $\lambda = 0$, we have

$$\frac{dEu}{d\theta}|_{\overline{\theta}} = -[1 - F(\overline{\theta})]F(\overline{\theta})u'(W - P(\overline{\theta})) + F(\overline{\theta}) \int_{\overline{\theta}}^{L} u'(W - P(\overline{\theta}) - x + \overline{\theta})f(x)dx$$

$$> -[1 - F(\overline{\theta})]F(\overline{\theta})u'(W - P(\overline{\theta})) + F(\overline{\theta}) \int_{\overline{\theta}}^{L} u'(W - P(\overline{\theta}))f(x)dx = 0. \quad (3)$$

The inequality in (3) follows from the fact that $u'(W-P-x+\overline{\theta}) > u'(W-P)$ for all $x > \overline{\theta}$. Since expected utility is increasing for every $\overline{\theta} < L$, it follows that full insurance, $\theta^* = L$, is optimal whenever the insurance premium is actuarially fair. This proves the first part of Mossin's Theorem. However, by our comments in the previous section, this optimum is not unique: note that any $\widehat{\theta} > L$ will also yield a maximum utility.

For a positive premium loading, $\lambda > 0$, consider

$$\frac{dEu}{d\theta}|_{\overline{\theta}} = -(1 + \lambda)[1 - F(\overline{\theta})]Eu'(\widetilde{y}) + \int_{\overline{\theta}}^{L} u'(W - P(\overline{\theta}) - x + \overline{\theta})f(x)dx. \quad (4)$$

Note that for $\overline{\theta} = L$, (4) seems to imply a value of zero. The problem is that $dEu/d\theta$ does not exist at $\overline{\theta} = L$. More explicitly, the derivative $\frac{dEu}{d\theta}|_{\theta=L}$ from the right is zero; however we are concerned with the derivative from the left. This can be obtained in a straightforward manner using L'Hôpital's rule. Restrict $\overline{\theta} < L$ and consider the limit of (4) as $\overline{\theta} \to L^-$. Since $\overline{\theta} < L$, it follows that

$$sgn\left\{\frac{dEu}{d\theta}|_{\overline{\theta}}\right\} = sgn\left\{\left(\frac{1}{1 - F(\overline{\theta})}\right)\left(\frac{dEu}{d\theta}|_{\overline{\theta}}\right)\right\}. \quad (5)$$

Taking limits as $\overline{\theta} \to L^-$ and applying L'Hôpital's rule,

$$\lim_{\overline{\theta} \to L^-} \left(\frac{1}{1 - F(\overline{\theta})}\right)\left(\frac{dEu}{d\theta}|_{\overline{\theta}}\right) = \lim_{\overline{\theta} \to L^-} -(1 + \lambda)Eu'(\widetilde{y}) +$$

$$\frac{-1}{f(\overline{\theta})}[-f(\overline{\theta})u'(W-P(\overline{\theta}))+(1-P'(\overline{\theta}))\int_{\overline{\theta}}^{L}u'(W-P(\overline{\theta})-x+\overline{\theta})f(x)dx]$$

$$= \quad -\lambda u'(W-P(L))<0. \tag{6}$$

It follows from (5) that the derivative in (4) must be negative for $\overline{\theta} \to L^-$. Hence $\theta^* < L$. Mossin's Theorem follows immediately.

Remark: Note that we do not need to worry about the second-order conditions for this proof. Since expected utility is constant for all $\theta \geq L$, we know that it is impossible to have $\theta^* > L$. Hence, it suffices to show that some values of $\theta < L$ yield a higher expected utility than $\theta = L$.

3 And for Deductibles?

Since the indemnity payoff structure for an upper-limit policy is somewhat symmetric to that for a deductible policy, one might wonder why we do not encounter the same fundamental problem in obtaining Mossin's Theorem for insurance contracts restricted to the class of deductibles. Actually there is a similarity, but also one important difference.

To see this one must first consider the notion of a negative level of deductibility. What would it mean if I were to choose a deductible of, say, negative $100 for my insurance policy? Does this make any kind of sense? It does if we define the deductible payoff as simply $I(x) = \max(x - D, 0)$, where D denotes the deductible level. For $D = -100$, this would be equivalent to $I(x) = x + 100$ for all x. In other words, a negative deductible of $D = -100$ is essentially a full insurance contract that also adds $100 for every loss amount, including a loss of zero. That is, it is a full insurance contract bundled together with a sure cash payment of $100.[2]

The important difference that I alluded to at the start of this section has to do with pricing. Whereas an upper limit that exceeds L adds no extra premium, a negative deductible will always increase the premium. Indeed, in our static model, the increase in premium to go from $D = 0$ to $D = -100$ will be $\Delta P = (1+\lambda)100$. Thus, for $\lambda > 0$, I will need to pay an addition $\$(1+\lambda)100$ to receive an extra $100 – hardly worthwhile. Thus, it is never optimal to have a negative deductible if $\lambda > 0$.

Of course, if we have a fair premium ($\lambda = 0$), then $\Delta P = 100$; or more generally, for a negative deductible of $-D$, the extra premium over full insurance is D. Obviously, in this static model, I am indifferent between any negative deductible and full insurance ($D = 0$). Thus, we obtain a non-unique optimum of full coverage in the case of fair insurance, similar to the result for upper-limit policies.

[2] Of course, a contract defined this way violates the so-called principle of indemnity: $I(x) \leq x \;\; \forall x$, but we are only looking at a mathematically feasible definition here. Negative deductibles do not actually exist to my knowledge.

4 Concluding Remarks

Mossin's Theorem, that partial insurance is optimal when insurance prices contain a positive proportional premium loading, whereas full coverage is optimal under a fair premium, was reexamined for the case of upper-limit insurance policies. The result holds for this class of policies, although the proof is less direct than that for deductible policies.

For the case of full coverage under a fair premium, I showed that the optimal contract (optimal upper limit, or optimal deductible level) is not unique. Upper limits exceeding the maximum loss and negative deductible levels do just as well. Of course, real-world policies are likely to restrict upper limits and deductibles to be within the interval $[0, L]$.

For a premium of the form $P = (1 + \lambda)E[I(\widetilde{x})]$, Pareto-efficient indemnity contracts are of the deductible form only in limited circumstances, and only in unusual circumstances might they include an upper limit or be linear in the level of damages, as examined carefully by Raviv (1979). In general, the optimal contractual form is likely to include more complex risk sharing.

Still, Mossin's Theorem seems so basic that we might postulate that it should hold for any class of insurance contracts. When $I(x)$ is either linear or a pure deductible or a pure upper-limit contract, the optimal level of coverage is easy to measure via just one parameter. That is, the coinsurance level or the deductible level of the upper limit is a parametric choice variable that indicates the level of coverage. In the more general case where $I(x)$ is the Pareto-efficient contract or even in cases where $I(x)$ depends on more than one choice parameter, for example choosing a deductible together with a linear rate of coinsurance for losses above the deductible, we lose the simplicity of having one parameter as a metric for "the level of coverage." In such instances, we would need to use the actuarial value $E[I(\widetilde{x})]$ to measure the level of coverage for any restricted form of indemnity functions $I(x)$.

I am not aware of any generalized formulation of Mossin's result in this direction. But, I do hope that the results presented here help one to obtain a deeper understanding of the nuances involved in moving the theoretical foundations of insurance demand in this direction.

References

Mossin, J., 1968, Aspects of Rational Insurance Purchasing, *Journal of Political Economy,* 76: 533-568.

Raviv, A., 1979, The Design of an Opitmal Insurance Policy, American Economic Review, 69: 84-96.

Schlesinger, H., 2000, The Theory of Insurance Demand, in G. Dionne, ed., *The Handbook of Insurance*, (Boston: Kluwer Academic Publishers) 131-151.

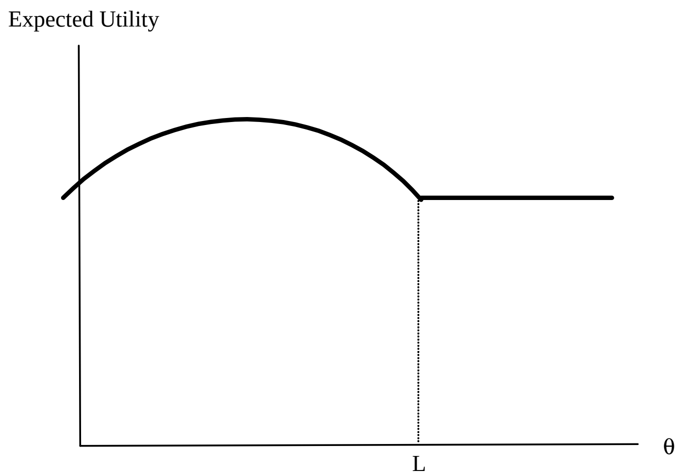

FIGURE 1