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Insurance Demand Without the Expected-Utility Paradigm

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Introduction

Risk aversion has always been the crucial element in generating a theory of insurance demand. Most especially, since the theory of risk aversion was made precise by Pratt (1966) and by Arrow (1971), a theory of insurance demand has been based upon this concept as enveloped within the expected-utility paradigm. Even more exacting, these theories have been developed within an expected-utility framework in which the underlying utility function is at least twice differentiable. This latter assumption is not innocuous. Without it, even some of the most basic insurance results, such as Mossin's Theorem ("full insurance is optimal under a proportional premium loading if and only if the price is fair") may fail to hold.

In this article, I examine two cornerstone results in the theory of insurance demand: Mossin's Theorem (Mossin (1968)) and Arrow's Theorem extorting the optimality of a straight deductible policy (Arrow (1971)). It turns out that Arrow's Theorem is very robust indeed, holding under risk aversion within any decision-theoretic framework. However, the "only if" part of Mossin's Theorem need not hold in general. As it turns out, the "culprit" for invalidating Mossin's result is not the lack of an expected-utility framework, but a lack of sufficient "smoothness" of preferences. Indeed, Mossin's theory cannot be extended to utility that is nondifferentiable at some wealth levels, even within the expected-utility framework.

The crucial element in extending Mossin's Theorem to risk-averse preferences is that risk aversion be of order 2, as defined by Segal and Spivak (1990). An extension of Mossin's Theorem to nonexpected-utility models can be found in the extensive treatment by Machina (1995). However, Machina assumes implicitly that risk aversion is of order 2 in his analysis, a point brought out by Karni (1995). If, however, risk aversion is of order 1, Mossin's result must be modified for it to apply to all risk-averse preferences. In particular, with risk aversion of order 1, full insurance may still be optimal with a positive premium loading, as was originally shown by Segal and Spivak (1990).

I also examine here whether or not the addition of an independent background risk will alter either Arrow's Theorem, or the modified version of Mossin's Theorem. Although the contribution of this paper is mainly pedagogical, the results on background risk are new; albeit they are quite easy to prove if one sets up the model correctly.

The extension of these insurance results beyond expected-utility models is not just a theoretical whim. Exceptions to the expected utility model, both experimental and empirical, have long been recognized. Since the purpose here is not to support or detract from the expected-utility model, interested readers are referred to Hershey and Schoemaker (1980) and to Machina (1987). However, many new positive theories have developed in the past few years, all of which are competing as an explanation to decision making under undertainty. In this paper, the two key insurance results named above are examined without the use of any particular model. Rather, they are extended to the collection of all models exhibiting risk aversion, which is defined here as an aversion to mean-preserving spreads, as defined by Rothschild and Stiglitz (1970). As a consequence, these results are "model independent" --they apply across models.

It is hoped that this more general approach also will lead to a better appreciation for the robustness of these results, all of which were devolped within the confines of expected utility theory. It is often the case that, if one is not convinced that a particular theory is beyond reproach, one tends to disregard results stemming from that theory. But, to put a new twist on Cleopatra's famous action: this is somewhat similar to destroying the message because you do not like the messenger. It is easy to find fault within competing models, such as the expected-utility model. Its axioms have been challenged and some of its predictions upset in simple laboratory experiments. Still, it has an established tradition in the theory of insurance economics and very few of the results derived within it depend on the entire complexity of the theory. Hopefully, any imperfections in the expected-utility messenger are shown here not to be sufficient grounds for ignoring its messages.

I begin in the next section with an introduction to orders of risk aversion. Here, as throughout the article, I simplify the theory to the extent that its use in the insurance models is unaffected. I apologize in advance to many of the original authors for over-simplifying their results. On the basis that more "tools" are better than fewer, I next show how orders of risk aversion affect Mossin's Theorem in a simple mean-variance framework. Such a framework is actually more general than one might realize, and it always coincides with expected-utility for decisions concerning (proportional) co-insurance, as was shown by Sinn (1980) and by Meyer (1989). I next examine the same result within a standard state-claims framework. Arrow's Theorem on the optimality of deductibles is then examined. Finally, all results are extended to include an uninsurable background risk.

Risk Aversion

The mainstay of any theory of insurance demand is risk aversion. An individual is defined to be *risk averse* if he or she dislikes any mean-preserving spread in the distribution of final wealth. If it is also assumed that an individual has a strict preference for more wealth under certainty, then preferences are consistent with second-degree stochastic dominance. In particular, if $\tilde{w}_1$ and $\tilde{w}_2$ denote two random variables representing final wealth, with corresponding distribution functions $F_1(w)$ and $F_2(w)$ respectively, then an individual is risk averse if

(1) $$F_1 \ SSD \ F_2 \ \Rightarrow \ \tilde{w}_1 \succsim \tilde{w}_2,$$

where "F_1 SSD F_2" denotes that F_1 second-degree stochastically dominates F_2 and the symbol "$\succsim$" denotes weak preference (i.e."is at least as good as").[1] Note that, by taking F_1 as a degenerate distribution giving the mean of F_2 with certainty, it follows from (1) that any risk averse individual always prefers receiving the mean of any wealth distribution with certainty to the distribution itself.

For any random wealth $\tilde{w}$, one can write

(2) $\tilde{w} = E(\tilde{w}) + \tilde{x},$

where "E" denotes the expectation operator and where $\tilde{x}$ is a random variable with mean zero, $E(\tilde{x}) = 0.$ To keep the presentation simple, suppose at first that $E(\tilde{w}) = 0.$ From (1) and (2) it follows that receiving zero with certainty is preferable to receiving the random amount $\tilde{x}$:

(3) $0 \succsim \tilde{x}.$

Assuming preferences for wealth are continuous, one can define a monetary amount $k > 0,$ such that adding k to $\tilde{x}$ would make the individual indifferent to receiving zero versus $\tilde{x} + k$:

(4) $0 \sim \tilde{x} + k$,

where "~" denotes indifference. Thus, k denotes a type of risk premium for $\tilde{x}$.[2]

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[1]Second-degree stochastic dominance of F_1 over F_2 is defined as $\int_{-\infty}^{y} [F_2(z) - F_1(z)]dz \geq 0 \ \forall y.$ This is equivalent to defining $\tilde{w}_2 =_d \tilde{w}_1 + \tilde{\varepsilon},$ where "$=_d$" means "equal in distribution" and where $E(\tilde{\varepsilon}|w_1) \leq 0 \ \ \forall w_1$. Note that if $\tilde{w}_1$ and $\tilde{w}_2$ have the same mean, i.e. $E(\tilde{\varepsilon}|w_1) = 0 \ \forall w_1$, then *SSD* is equivalent to a sequence of mean-preserving spreads (see Rothschild and Stiglitz (1970)). The definition of risk aversion used above seems to be the most generally accepted, but alternatives to (1) could lead to alternative conclusions. A comprehensive survey and analysis can be found in Cohen (1995).

[2]Note that this definition is not quite the same as Pratt's (1964) risk premium π, which is defined via $\tilde{x} \sim 0 - \pi.$ In other words, rather than adding wealth to the right-hand side of (3) as we do in (4), the Pratt risk premium subtracts from the left-hand side of (3). This distinction is essentially that between

In a recent important paper, Segal and Spivak (1990) make a distinction between two categories of risk aversion, which they label as first- and second-order risk aversion. To define these concepts, let $t > 0$ and define $k(t)$ as the risk premium for $t\tilde{x}$:

(5) $$0 \sim t\tilde{x} + k(t).$$

Since preferences are continuous, $k(t)$ will be continuous, and clearly $k(t) \to 0$ as $t \to 0^+$. (Recall that we restrict $t > 0$. The notation "$t \to 0^+$" means that t approaches zero from the positive side.) I also assume that $k(t)$ is differentiable for $t > 0$. Preferences are said to be risk averse of order 1 if $k'(t) \to \varepsilon > 0$ as $t \to 0^+$, and are said to be risk averse of order 2 if $k'(t) \to 0$ as $t \to 0^+$. That is,[3]

(6) $$\lim_{t \to 0^+} k'(t) = \begin{cases} \varepsilon > 0 & \text{First Order Risk Aversion} \\ 0 & \text{Second Order Risk Aversion} \end{cases}$$

If risk aversion is of order 2, then at the limit, when the amount of risk is infintessimal, the individual behaves in a risk-neutral manner. For example, risk-averse expected-utility preferences with differentiable utility satisfy second-order risk aversion. As is well known (see Arrow (1971)), individuals with such preferences will always accept some positive fraction (which might be quite small) of any gamble that has a positive expected payoff. Individuals that are first-order risk averse, by contrast, will find the expected positive payoff on some gambles to be too small to accept any fraction of the gamble, no matter how small. Such individuals behave in a risk-averse manner at the margin, even when the initial risk is zero. Whether or not real-world preferences satisfy first- or second-order risk aversion is an empirical question.

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compensating and equivalent variation in consumer choice under certainty. Using k as defined above is done for convenience and what follows could easily extend from Pratt (1964) instead.

[3]Segal and Spivak themselves use a somewhat more general setting regarding risk attitudes, and thus have a slightly more complex definition.

To extend these concepts to nonzero-mean risks, assume now that $E(\tilde{w}) \equiv \mu$. One can define the risk premium for $t\tilde{x}$ conditional on μ as $k(t;\mu)$, satisfying

(7) $\mu \sim \mu + t\tilde{x} + k(t;\mu).$

The properties of first-order and second-order risk aversion are local properties, and an individual's type of risk aversion may be of the first-order for some levels of μ and of the second-order for other levels of μ. For example, most all results on insurance economics under expected-utility theory assume that the utility function is everywhere differentiable. However, if the utility function is assumed to be concave (i.e. risk averse) and continuous, but if "kinks" are allowed, risk aversion will be of order 2 everywhere except at the kinks, where it will be of order 1.

To make things concrete, I examine only the cases where preferences are globally of order 1 or of order 2, and I label these types of preferences RA-1 and RA-2 respectively.

Risk Aversion under Mean-Variance Preferences

Before proceeding to the insurance results, I would like to illustrate the effects of RA-1 and RA-2 in a mean-variance setting. The generality of this setting is actually greater than many people realize, as I explain in the next section.

Consider the standard two-parameter preference functional, which depends on only the mean and standard deviation of the distribution function:

(8) $v(F) = u(\mu, \sigma),$

where $\mu = E(\tilde{w})$ and σ denotes the standard deviation of $\tilde{w}'s$ distribution. Preferences are assumed to be increasing in certain wealth and risk averse, which implies that

(9) $\frac{\partial u}{\partial \mu} > 0$ and $\frac{\partial u}{\partial \sigma} < 0$.

Preferences also are assumed to exhibit a preference for diversification.[4] Indifference curves for the above preferences are illustrated in Figure 1. Indifference curves in this figure are upward-sloping due to risk aversion and convex due to a preference for diversification. Thus, given any two random wealth distributions that are equally desirable, such as $\tilde{x}$ and $\tilde{z}$ in Figure 1, the consumer prefers any convex combination of $\tilde{x}$ and $\tilde{z}$, illustrated by the shaded triangular region in the figure, to either $\tilde{x}$ or $\tilde{z}$ alone.[5]

Figure 2 illustrates the risk premium $k(t)$ for $t = 1$ and for $0 < t < 1$. As t decreases to zero, we see in Figure 2 that $k(t)$ also decreases to zero. The condition that $k'(t) \to 0$ as $t \to 0^+$ implies that the indifference curve is flat (i.e. has a zero slope) when $\mu = \sigma = 0$. If preferences are RA-2, then all indifference curves have a zero slope when $\sigma = 0$. This is true for instance, if the preferences shown are consistent with (differentiable) expected-utility preferences. RA-2 also holds for preferences that are modeled by smooth local utility functions, as described by Mark Machina (1982). The theory of Machina's has advantages over von Neuman-Morgenstern utility theory in that it bypasses some long standing problems in the expected utility arena, such as the famous Allais Paradox [see Machina (1987)].

If preferences exhibit RA-1, indifference curves will all have a positive slope when $\sigma = 0$. Examples of some recent non-expected utility models exhibiting first-order risk

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[4]Assuming a preference for diversification is not an innocuous assumption in general. However, relaxing this assumption will not change any of the results in this article. I address this assumption in the next section.

[5]Since correlations of the wealth distributions are not apparent in the μ–σ diagrams, the set of possible convex combinations, or "portfolios," of $\tilde{x}$ and $\tilde{z}$ lies somewhere in the shaded triangular area in Figure 1. The rightmost edge of this triangle corresponds to possible portfolios when $\tilde{x}$ and $\tilde{z}$ are perfectly positively correlated. The other two edges correspond to a perfect negative correlation. Preference for diversification implies that all of these portfolios are strictly preferred to $\tilde{x}$ or $\tilde{z}$ alone. Note that preference for diversification is equivalent to risk aversion in expected utility models but does not always follow from risk aversion in non-expected utility models. See Deckel (1989) for further discussion.

aversion are the "dual theory" of Menachim Yaari (1987) and the "anticipated utility theory" of John Quiggin (1982).[6] These and many other non-expected utility theories have made headway into the literature on decision making under uncertainty. As we shall soon see, many results in the literature on insurance economics, as developed under expected utility, hold under RA-2 but must be modified under RA-1.

The Optimality of Full Versus Partial Coverage

Assume that an individual has an initial wealth of amount $A > 0$, which is subject to a random loss of amount $\tilde{L}$. Insurance is available, which pays out the indemnity $I(L)$ when the realized loss is L, for a premium $P[I(L)]$. In particular, consider the fairly common case in the literature in which $I(L) = \alpha L$, where α is chosen by the insured, $0 \leq \alpha \leq 1$, and where the premium for partial coverage is proportional to the full-coverage premium. Thus, the individual's wealth prospect is:

(10) $$\tilde{w}_\alpha = A - \tilde{L} - \alpha P_f + \alpha \tilde{L} = (A - P_f) + (1-\alpha)(P_f - \tilde{L}),$$

where P_f denotes the premium for full coverage:

(11) $$P_f = (1+\lambda)E(\tilde{L}), \quad \lambda \geq 0.$$

The coefficient λ is the premium loading factor for profit and expenses.

From (10) it is clear that the set of all possible final wealth distributions, one for each possible choice of α, differ from one another only by a shift in the mean and a proportional stretching. This allows us to use an important result from Hans-Werner Sinn (1980), which also was discovered independently by Jack Meyer (1987).

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[6]Quiggin's theory has been re-labeled in most of the literature, and today typically is called "Expected Utility with Rank-Dependent Preferences," or "Rank-Dependent Expected Utility."

***Sinn's Theorem**: If the choice set of alternative final wealth random variables differ from one another only via a change in location and scale, then any choices modeled using expected utility can be duplicated using a preference functional which depends upon only the means and standard deviations of the alternatives.*

This is a quite powerful result. For the case of proportional insurance, as given in (10), it says that we can always find $\mu-\sigma$ preferences that are consistent with expected-utility rankings of the different levels of insurance coverage.[7]

To examine how insurance results might or might not extend from expected utility to other preference models, consider first a cornerstone result in modern insurance economics, which is due to Jan Mossin (1968):

***Mossin's Theorem:** A risk-averse expected-utility maximizer buying proportional insurance coverage will choose*

- *(i)* *full coverage* $(\alpha^*=1)$ if $\lambda=0$.
- *(ii)* *partial coverage* $(\alpha^*<1)$ if $\lambda>0$.

Mossin's Theorem is illustrated in Figures 3 and 4. In Figure 3, the point $A-\tilde{L}$ denotes the initial random wealth position without insurance. Under an actuarially fair premium with $\lambda=0$, the expected value of final wealth is unaffected by the level of insurance and only the standard deviation changes. The wealth level $A-P_f$ with certainty $(\sigma=0)$ represents full insurance coverage, $\alpha=1$. The $\mu-\sigma$ final wealth

[7] A word of caution is in order, however. Sinn's Theorem does not imply that the $\mu-\sigma$ preferences and expected-utility preferences would agree for decisions made outisde of the choice set. For example, deductible insurance would not be subject to Sinn's Theorem. It is also worth noting that a recent paper by Ormiston and Quiggin (1993) extends the Theorem to include certain deviations from simple changes in scale; namely they allow for monotone mean-preserving spreads, which is a bit more general. They also extend the Theorem to models using Expected Utility with Rank-Dependent Preferences.

positions located along the line segment from $A-\tilde{L}$ to $A-P_f$ in Figure 3 represent final wealth corresponding proportionally to insurance levels ranging from $\alpha=0$ to $\alpha=1.$

Since the line segment of final-wealth locations is flat, it is seen that full insurance will be purchased whenever risk aversion holds, regardless of whether it is RA-1 or RA-2. Thus, $\alpha^*=1$ is the optimal insurance level, as is seen in Figure 3. Since risk aversion was the only relevant property used in obtaining this result, part (i) of Mossin's Theorem is easily seen to extend to all risk-averse preferences, regardless of whether risk aversion is of order 1 or of order 2.[8]

Now assume that there is a strictly positive proportional premium loading $\lambda>0.$ In this case, expected final wealth falls with the purchase of higher levels of insurance, trading off against the lower levels of risk. Thus, the set of final wealth locations after the purchase of insurance are located along the line segment from $A-\tilde{L}$ to $A-P_f$ in Figure 4, which now exhibits a positive slope. If risk aversion is of order 2, the individual will always be better off with something less than full insurance coverage. For example, $0<\alpha^*<1$ in Figure 4, with the optimal final-wealth location at point B. Thus, part (ii) of Mossin's Theorem extends to any model in which risk aversion is of order 2, such as Machina's (1982) model of "smooth" local utility functions.

Consider, however, the case $\lambda>0$ when preferences are RA-1. In this case, it is possible that the positive slope of the indifference curve through the point $A-P_f$ (with $\sigma=0)$ is steeper than the slope of the line segment depicting all of the final wealth alternatives. Such a case is illustrated in Figure 5. When this occurs, the individual is better off with full insurance than with any other insurance contract, thus violating part (ii) of Mossin's Theorem. In other words, full insurance might be purchased, even with a

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[8]Actually, the illustration holds only for $\mu-\sigma$ preferences and for expected utility or expected utility with rank-dependent preferences via Sinn's Theorem. The general case of (i) follows trivially from the fact that reductions in α are mean-preserving increases in risk and hence less preferred by all risk averters defined via (1).

positive premium loading, if risk aversion is of order 1. This result coincides with remarks attributable to Karl Borch (1990, p. 32):

> "We would indeed be surprised if a traveler deliberately
> insured his baggage for less than its full value . . . "

RA-1 is a possible explanatory factor for Borch's observation.

Note that the definition of risk aversion (1) guarantees that part (i) of Mossin's Theorem holds (see footnote 8) for all risk averters. Preferences may be either RA-1 or RA-2 and this result does not make use of the assumption concerning preference for diversification. Indeed, in the $\mu - \sigma$ illustration, the lack of a preference for diversification would imply that indifference curves are upward sloping but not necessarily convex. This would allow for multiple optima when insurance has a positive loading, $\lambda > 0$. It would also allow for an unrestricted optimum of $\alpha^* = -\infty$ when $\lambda > 0$.[9] This case yields a restricted optimum of no insurance, $\alpha^* = 0$. But these oddities do not affect the results of Mossin's Theorem.

In summary, assuming only risk aversion (of either order), the following is a more robust version of Mossin's results:

Modified Mossin's Theorem: *A risk-averse individual (not necessarily an expected-utility maximizer) buying proportional insurance coverage will choose*

(i) *full coverage* $(\alpha^* = 1)$ *if* $\lambda = 0$.

(ii) *full coverage* $(\alpha^* = 1)$ *or partial coverage* $(\alpha^* < 1)$ *if* $\lambda > 0$.

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[9]Note that $\alpha < 0$ is "going short" on losses. Allowing $\alpha = -\infty$ is what Tobin (1958) refers to as plunging (see also Deckel (1989)).

Although this modified version of Mossin's results is somewhat weaker, it holds for risk-averse preferences of either order, and also seems to fit somewhat better with casual empirical observations.

A State-Claims Analysis of Mossin's Theorem

The results of the previous section are readily illustrated using familiar state-claims diagrams for the case where $\tilde{L}$ has a two-point support; that is, for the case where $\tilde{L}$ takes the value $L_o > 0$ with probability *p*, and $\tilde{L}$ takes the value zero with probability *1-p*. Indeed, this type of a setting was examined by Yaari (1969) without the assumption of expected-utility. More recently, Machina (1995) provided an excellent overview of this framework in a nonexpected-utility setting. However, these authors assumed preferences were smooth, implicitly ruling out the possibility of RA-1.

The basic state-claims diagram of Yaari (1969) has become a useful tool in insurance economics, most typically in an expected-utility framework (see Hirshleifer (1965), (1966) and Kahane, Schlesinger and Yanai (1988)). In Figure 6, the line segment $\overline{BD}$ is given by the equation $(1-p)y_{NL} + py_L = \mu$, where (y_{NL}, y_L) represents the individual's wealth contingent on the states of nature "no loss" and "loss" respectively. The point C represents the certain wealth of μ, whereas other points on this line segment yield the same expected wealth as receiving μ for certain. Movements from C towards B or from C towards D are easily shown to be mean-preserving increases in risk as defined by Rothschild and Stiglitz (1970). Consequently, any risk-averse individual will be made worse off by such movements. Thus, C is the most-preferred contingent claim on segment $\overline{BD}$. This should come as no surprise since definition (1) implies preference for certainty over uncertainty with equal means.

So choose any initial endowment along $\overline{CD}$, such as E, which implicitly defines W and L_o and viola! Mossin's Theorem part (i) holds. Indeed, increases in the level of coverage α imply movements from E towards C along $\overline{BD}$. Thus, without even drawing any indifference curve we know that C is the optimal contingent claim, which of course is obtained via full insurance, $\alpha^* = 1$.

Given the above arguments, any indifference curve through C must lie completely above segment $\overline{BD}$ except, of course, at C itself. Since preferences are monotonic (i. e. more wealth is preferred to less) indifference curves are downward sloping. If preferences are also smooth, then the indifference curve must be locally convex at C with the same slope as $\overline{BD}$, namely $-(1-p)/p$. However, preferences need not be globally convex under risk aversion, even under RA-2, if we depart from the expected-utility model. But note here that global convexity is not needed to show that full coverage is optimal.

To illustrate the effect of RA-1, define the random variable $\tilde{x}$ as the contingent claim with a payoff of $-\$(1-p)$ in the loss state, and a payoff of $+\$p$ in the no loss state. Thus, $E\tilde{x} = 0$. Now consider the random variable $t\tilde{x}$ added to μ for $t > 0$. From the definition of the conditional risk premium given in (7), one can substitute for $\tilde{x}$ to find the state claim indifferent to (μ, μ):

(12) $$(\mu,\mu) \sim (\mu + k(t;\mu) + tp,\ \mu + k(t;\mu) - t(1-p)).$$

The slope joining the two state claims given in (12) is seen to be

(13) $$\text{slope } = -\frac{t(1-p) - k(t;\mu)}{tp + k(t;\mu)}.$$

To find the slope from the right of the indifference curve through point C in Figure 6, take the limit of (13) as $t \to 0^+$. Using L'Hôpital's rule, this yields

(14) $$\lim_{t\to 0^+} -\frac{t(1-p)-k(t;\mu)}{tp+k(t;\mu)} = -\frac{(1-p)-k'(t;\mu)}{p+k'(t;\mu)}\Big|_{t=0^+}$$

If risk aversion is of order 2, this slope equals $-(1-p)/p$ as discussed above. However, if risk aversion is of order 1, the slope from the right will be flatter than $-(1-p)/p$. Replacing $\tilde{x}$ with $-\tilde{x}$, a similar argument shows that the slope from the left will be steeper than $-(1-p)/p$ under RA-1. An illustration is given in Figure 6 with the indifference curve ICI' passing through (μ,μ).

If there is a preference for diversification, indifference curves will each be strictly convex. Without a preference for diversification, the indifference curves have no local convexity or concavity constraints, except they are strictly convex where they cross the certainty line. Machina (1995) provides a useful and simple analysis of such preferences under RD-2, so I will not spend time on details here. The RD-1 indifference curve as drawn in Figure 6, is linear on either side of the certainty line; which indicates an indifference to diversification, unless diversification changes the preferred state.[10]

Actually, the preferences illustrated in Figure 6 correspond to a particular nonexpected utility model, namely the dual theory of Menachin Yaari (1987).[11] These preferences are used next to illustrate part (ii) of Mossin's Theorem.

Suppose there is a positive proportional premium loading for the insurance scenario told for Figure 6. For P_f given by (11) with $\lambda > 0$, it follows easily that the

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[10]State claims lying on the same side of the certainty line are said to be "co-monotonic." The preferences in Figure 6 illustrate indifference to differsifying across co-monotonic claims.

[11]Rather than spending time on particular models, I prefer staying in the more general setting. However, a few remarks might help here. Yaari's model is a special case of Rank-Dependent Expected Utility as developed by Quiggin (1982). It is obtained by replacing the distribution function with a transformation of itself. Thus let $g:[0,1]\to[0,1]$ be a bijective, monotone increasing function. Let $u(\cdot)$ be a utility function and $\tilde{y}$ a random variable with distribution $F(y)$. Preferences under RDEU are expressed as

$$V(F) \equiv \int_{-\infty}^{+\infty} u(y)d[g(F(y))].$$

The property of risk aversion (1), is satisfied if and only if both *u* and *g* are concave (one strictly concave for strict risk aversion). Yaari's dual theory is the special case of RDEU where $u(y) = y$. Risk averse RDEU preferences are easily shown to be of order 1 if *g* is strictly concave, and of order 2 if *g* is linear. See Quiggin (1993) for an ecompassing treatise on RDEU preferences.

insurance opportunities lie along a line segment through the initial position E with a flatter slope than $\overline{BD}$; in particular with a slope $-[(1/(1+\lambda))-p]/p$. This is illustrated by the line segment $\overline{EG}$ in Figure 7, with insurance contracts restricted to lie along $\overline{EH}$. As drawn, clearly *H* is the most preferred state-claim along this segment, where *H* represents wealth under full insurance, $H=(A-P_f, A-P_f)$. Thus, full insurance is optimal, even though there is a positive premium loading, contrary to Mossin's original result.

Obviously, if the premium loading is high enough, the segment $\overline{EH}$ could be of a slope equal to or flatter than the indifference curve through *E*, indicating respectively an indifference to coverage levels α and a preference for no insurance coverage at all. In other words, using the preferences illustrated in Figures 6 and 7, one obtains a type of "bang-bang" solution, whereby either full insurance is purchased, or no insurance is purchased. A more detailed review of results in Yaari's model can be found in Doherty and Eeckhoudt (1995).

For both RD-2 and RD-1 preferences in general, the indifference curves will be everywhere convex if and only if there is a preference for diversification. When there is no such preference, multiple optima are possible. Although this does not alter the results obtained thus far, it is possible that part (ii) of the Modified Mossin's Theorem holds with both full coverage and a partial level of coverage being optimal. An illustration of such a case is presented in Figure 8. Once again claim E denotes the individual's original uninsured wealth position and insurance includes a positive loading, yielding potential final wealth claims along the segment $\overline{EH}$. As drawn in Figure 8, both full coverage at state claim *H* and partial coverage at state claim *J* are optimal.[12]

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[12]An analysis of comparative statics without expected utility is presented by Machina (1995). The possibility of multiple optima does not seem to be too great of a problem itself. However, risk aversion alone is not sufficient for many of the interesting comparative static results in expected utility models. Thus, it certainly is not sufficient in a more general setting. The additional restrictions needed become fairly model specific.

Optimality of Deductibles

A well-known result in insurance economics under expected utility is that the Pareto-optimal form of insurance indemnification is a deductible policy whenever (1) the insurer's costs are proportional to the indemnity payment and (2) the insurer is risk neutral. This basic result is due to Arrow (1971) and has been extended in numerous ways, most notably by Raviv (1979). A review of the literature is found in Gollier (1992).

Recently, Karni (1992) has shown that this result can extend to nonexpected-utility models that satisfy certain differentiability criteria. Karni, as well as Arrow and Raviv before him, relies on dynamic optimization techniques to solve for the optimal functional form of the indemnity function. However, any set of preferences that satisfy risk aversion as given in (1) can be shown to yield deductible policies as an optimum. Gollier and Schlesinger (1996) recently showed this result directly by applying SSD arguments.[13] This direct approach requires no knowledge of dynamic optimization techniques. Moreover, it provides insight into the basis for the optimality of deductibles. I provide below a sketch of the basic arguments. The details can be found in Gollier and Schlesinger (1996).

Consider a fixed premium P and fixed actuarial value for an insurance contract, $E[I(\tilde{L})]$, where $I(L)$ denotes the indemnity paid for loss L. It is assumed only that $I(\cdot)$ must be a nondecreasing function with $0 \leq I(L) \leq L$ for all values of L. By the assumptions used in Arrow's model, the expected profit level (and hence the welfare level) of the insurer is fixed. Thus, deductible insurance will represent an optimal contract if it is the most preferred type of contract for every risk averter, where risk aversion is defined by (1).

The individual's final wealth can be written as

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[13]This was shown indirectly by Zilcha and Chew (1990), who proved that Pareto-efficient outcome sets under expected utility are invariant when the comparison changes to any preferences satisfying (1).

(15) $\tilde{w} = A - \tilde{L} - P + I(\tilde{L}).$

In order to see that a deductible is most preferred, consider the pre-indemnity wealth position of the individual,

(16) $\tilde{w}_0 \equiv A - \tilde{L} - P.$

The wealth given by (16) is the individual's wealth immediately following a loss, having paid the premium but not yet having received an indemnity. An illustration of the distribution function, *F*, of $\tilde{w}_0$ is given in Figure 9. As drawn, $\tilde{w}_0$ and hence $\tilde{L}$ are continuous random variables, but this need not be the case to show the result.

Now add in the indemnity, $I(L)$, being certain that $I(\cdot)$ is monotone increasing with $0 \le I(L) \le L$ and with $E[I(\tilde{L})]$ fixed. Since $I(\cdot)$ is positive and monotone, it follows that only first-degree stochastic dominance changes to *F* are allowed when adding $I(L)$. In other words, we can only shift F downwards (i.e. shift probability mass to the right) and we do this until the mean of F increases by $E[I(\tilde{L})]$. One possible $I(L)$ is the deductible, which pays $I(L) = L - d$ for all $L \ge d$ and zero otherwise. The level *d* is determined here by $E[I(\tilde{L})]$, since the actuarial value is fixed.[14] The distribution function for wealth with a deductible is shown in Figure 10 as the discontinuous distribution *G*. This distribution simply shifts all probability mass between zero and *A-P-d* to a mass point at *A-P-d*.

Now any alternative final wealth distribution for a fixed $E[I(\tilde{L})]$ will have an equal mean to *G*. Since $I(L) \ge 0$ and since *G* and *F* are identical for wealth in the interval [*A*-

[14]Recall that we are not looking for an optimal level of the deductible, only for the optimal form of the contract for a fixed $E[I(\tilde{L})]$. If a deductible is optimal for every actuarial value and if the premium depends only on the actuarial value, then the optimal overall contract will be one of these deductibles.

P-d, A-P], any alternative indemnity function can only reduce G in this interval. But then, since the mean wealth must be preserved, this alternative must increase G in the interval $[0, A-P-d]$. These two types of changes, taken in tandem, are a mean-preserving increase in risk from G (i.e. an SSD worsening) and thus is disliked by all risk averters, as defined by (1). Consequently, the deductible policy is the most preferred, thus proving Arrow's Theorem.

Background Risks

The results presented thus far are based on a single source of risk, namely the random loss $\tilde{L}$. Ever since Doherty and Schlesinger (1983), many of the questions addressed here have been examined in the presence of multiple risks, mostly within an expected-utility framework, but on occasion within particular nonexpected-utility models.[15]

All of the results of which I am aware of, examining behavior in the presence of background risk, are modelled within the confines of a particular preference functional. But which results that have been presented in this article thus far are robust enough to apply to all models satisfying risk aversion as defined by (1), in the presence of background risk? Certainly all of the results can be violated if the background risk is not independent of $\tilde{L}$. So suffice it to say that interrelationships between risks have pervasive effects.[16] However, what of the case of an independent background risk? Does Mossin's Theorem still apply; and does Arrow's Theorem on the optimality of

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[15]A good example of the former is Eeckhoudt and Kimball (1992), who examine how a background risk can have qualitatively predictable affects on insurance demand if one is willing to assume conditions stronger than risk aversion. Gollier and Pratt (1996) further this analysis by determining conditions on utility that are both necessary and sufficient for an independent background risk to increase insurance demand. A nice example under nonexpected utility is provided by Doherty and Eeckhoudt (1995) in a model applying Yaari's (1987) dual theory. They show how an independent background risk can induce further convexity into preferences, resulting in a level of coverage between no coverage and full coverage, thus negating the "bang-bang" type of insurance demand that occurs without background risk.

[16]Two marvelous recent papers, Aboudi and Thon (1995) and Tibletti (1995), introduce some new statistical tools from probability theory, tools that measure statistical interrelationships, into the literature on insurance economics.

deductibles still hold? Doherty and Schlesinger (1983) and Gollier and Schlesinger (1995) have shown the answers to both questions to be "yes" within the confines of expected-utility theory.

It turns out to be quite easy to see that the answer to both questions is also yes when the expected-utility hypothesis is relaxed. Consider first Mossin's Theorem. Let insured wealth be given by equations (10) and (11) with the exception that initial wealth A is replaced by $A+\tilde{\varepsilon}$, where $E(\tilde{\varepsilon})=0$ and where $\tilde{\varepsilon}$ and $\tilde{L}$ are statistically independent. Thus, defining $\tilde{w}_\alpha$ as given in (10), final wealth is now $\tilde{w}_\alpha+\tilde{\varepsilon}$.

If the premium is actuarilly fair, $\lambda=0$, then full coverage was seen to be optimal without background risk. This followed since the "sure thing" of receiving $A-P_f$ second-order stochastically dominated every other possible $\tilde{w}_\alpha$ with $\alpha\neq 1$. Viewing $\tilde{w}_1$ as a degenerate random variable paying $A-P_f$ with certainty, one can write $\tilde{w}_1\ SSD\ \tilde{w}_\alpha$ for all $\alpha\neq 1$. But from stochastic dominance theory, it follows trivially that $\tilde{w}_1+\tilde{\varepsilon}\ SSD\ \tilde{w}_\alpha+\tilde{\varepsilon}$ for all $\alpha\neq 1$, and hence full coverage on $\tilde{L}$ is always optimal.[17] Thus part (i) of Mossin's Theorem holds true. Since one can always find RA-1 preferences and examples of $\tilde{\varepsilon}$ such that $\alpha^*=1$ and such that $\alpha^*<1$, whenever there is a premium loading, part (ii) of Modified Mossin's Theorem also holds.[18] Thus, the modified version of Mossin's Theorem holds in the presence of an independent background risk.

Turning to Arrow's Theorem, the extension is just as simple. Let $\tilde{w}_I$ denote wealth with insurance indemnity $I(\tilde{L})$, as given in equation (15). Let $\tilde{w}_d$ denote the special case where $I(\cdot)$ represents a deductible policy. Thus, for a fixed premium *P* and fixed actuarial value, $\tilde{w}_d\ SSD\ \tilde{w}_I$ for every other possible indemnity function $I(\cdot)$. But then it follows once again from stochastic dominance theory that $\tilde{w}_d+\tilde{\varepsilon}\ SSD\ \tilde{w}_I+\tilde{\varepsilon}$

[17]One way to see this is to invoke Rothschild and Stiglitz (1970), who shows that $\tilde{w}_1\ SSD\ \tilde{w}_\alpha$ is equivalent to the existence of a random variable $\tilde{y}$, such that $\tilde{w}_\alpha =_d \tilde{w}_1+\tilde{y},\ E(\tilde{y}|w_1)=0$, where "$=_d$" denotes "equal indistribution." Now since $\tilde{\varepsilon}$ is independent of $\tilde{w}_\alpha$ it must be independent of $\tilde{y}$. Thus $\tilde{w}_\alpha+\tilde{\varepsilon} =_d \tilde{w}_1+\tilde{\varepsilon}+\tilde{y}$ where $E(\tilde{y}|w_1+\varepsilon)=0$. Thus $\tilde{w}_1+\tilde{\varepsilon}\ SSD\ \tilde{w}_\alpha+\tilde{\varepsilon}$ for all $\alpha\neq 1$.

[18]See, for instance, the paper by Doherty and Eeckhoudt (1995).

for any $\tilde{\varepsilon}$ such that $\tilde{\varepsilon}$ and $\tilde{L}$ are independently distributed. Therefore a deductible policy is optimal and Arrow's Theorem holds.[19]

Concluding Remarks

With a modification to Mossin's Theorem, to account for RA-1, both this Theorem and Arrow's Theorem extend to nonexpected-utility models exhibiting risk aversion as defined by a preference for second-degree stochastic dominance. Moreover, the modification to Mossin's result - - that full coverage may be optimal even when there is a positive premium loading - - fits well with casual empiricism. The fact that risk aversion alone gives us these results, and not the particulars of the framework employed, is quite powerful. The fact that these basic results hold in all models exhibiting risk aversion makes them very robust indeed.

Of course, many of the interesting questions about the demand for insurance examine comparative static effects.[20] For example, what happens if the price of insurance increases (Mossin (1968)); what happens if initial wealth increases (Arrow (1971)); what happens to the level of partial insurance demanded with the addition of a background risk (Eeckhoudt and Kimball (1992))? All of these and a plethora of other interesting questions depend upon more than simply risk aversion. Indeed, within the confines of expected-utility models, each of these questions does not have a definitive qualitative answer, if only risk aversion is assumed. Thus, they cannot rely on risk aversion alone in the more general setting of nonexpected-utility preferences.[21]

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[19]Doherty and Schlesinger (1983) showed that a deductible need not be optimal if $\tilde{\varepsilon}$ and $\tilde{L}$ are not statistically independent. Gollier and Schlesinger (1995) proved that Arrow's Theorem holds with an independent background risk within an expected-utility framework.

[20]Another line of interesting questions examines insurance models of asymmetric information, such as moral hazard and adverse selection, which go beyond the scope of the current article. Some of these extensions are addressed in Schlesinger (1994).

[21]As pointed out by Machina (1995), it is not advisable to view nonexpected-utility theory as an alternative to expected-utility theory. Rather, one should view nonexpected utility as a generalization of the expected-utility approach.

A natural question then is whether or not the concepts required to answer each of these questions in an expected-utility setting can be generalized to apply outside of the expected-utility framework? Although much literature already exists on extending comparative risk aversion, much is still left to be done in extending concepts such as prudence (Kimball (1990)) and risk tolerance (Gollier and Pratt (1996)).

Moreover, experimental and/or empirical findings, such as a finding that full insurance is purchased by some individuals at a "loaded" price, can be better interpreted if one knows the robustness of the result. From the results presented here, this finding would tend to support risk aversion of order 1. However, even here one needs to be careful. The models presented in this paper are all based on preference functionals over final wealth distributions. If nonpecuniary elements affect preferences or if individuals satifice rather than optimize (as is bounded rationality models), then the models presented here will not apply. Moreover, pecuniary complications such as transactions costs and/or intertemporal wealth affects can also affect insurance demand.

While I make no claims to have found an all encompassing explanation of the demand for insurance in this article, I do hope the article has shown how we can work to improve upon what we already know.

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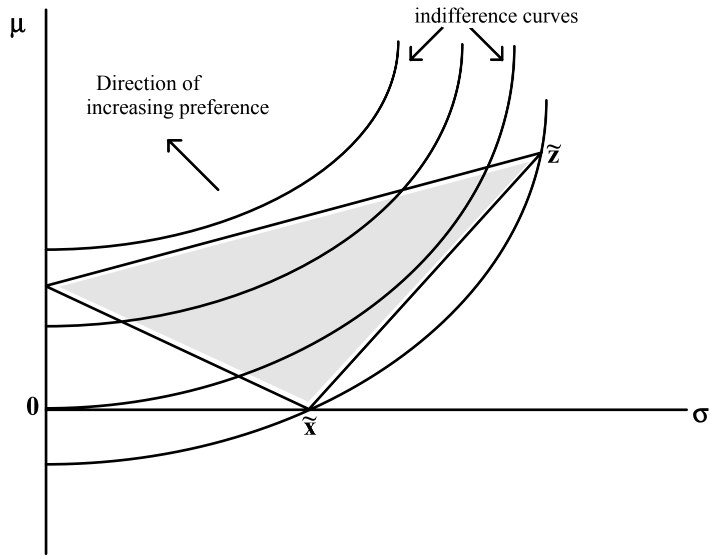

Figure 1 Risk aversion and a preference for diversification

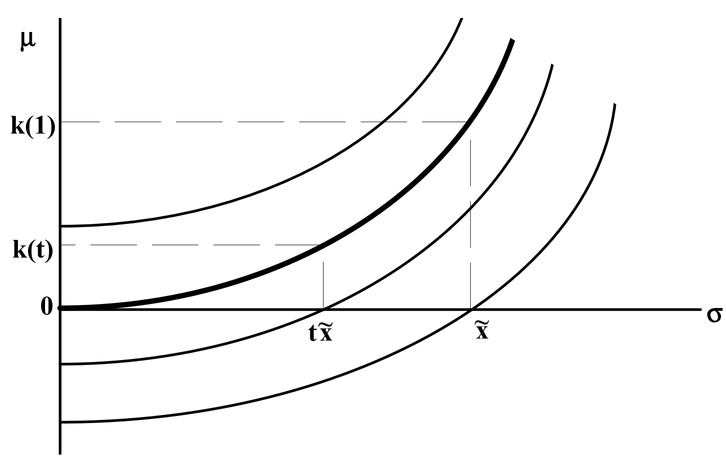

Figure 2 The risk premium k(t)

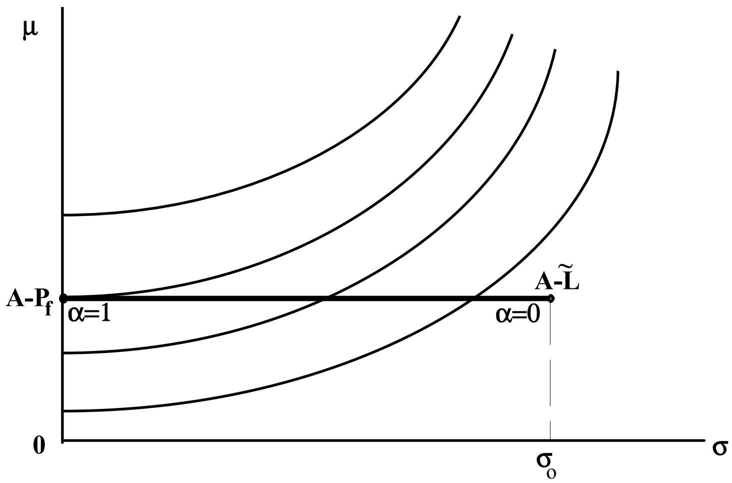

Figure 3 Mossin's Theorem: Full insurance at a fair price

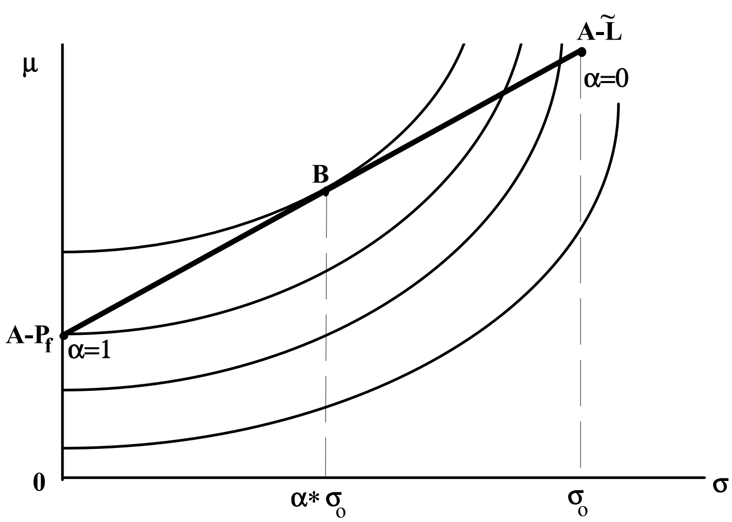

Figure 4 Mossin's Theorem: Partial insurance at a loaded price

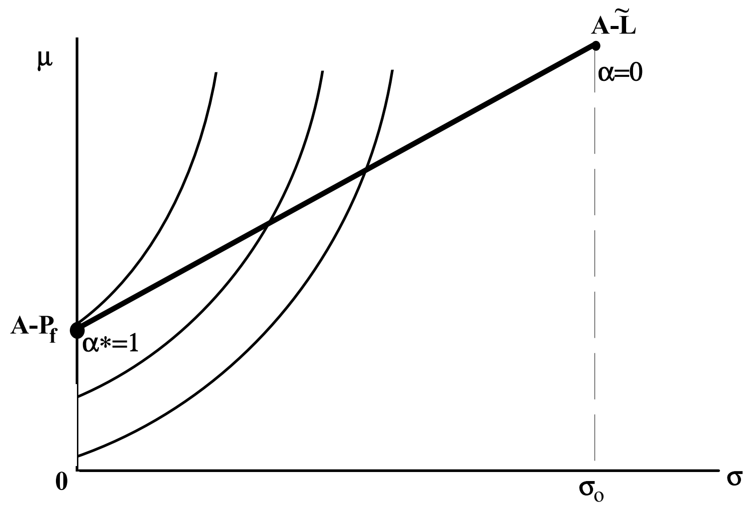

Figure 5 Modified Mossin's Theorem: Full insurance at a loaded price

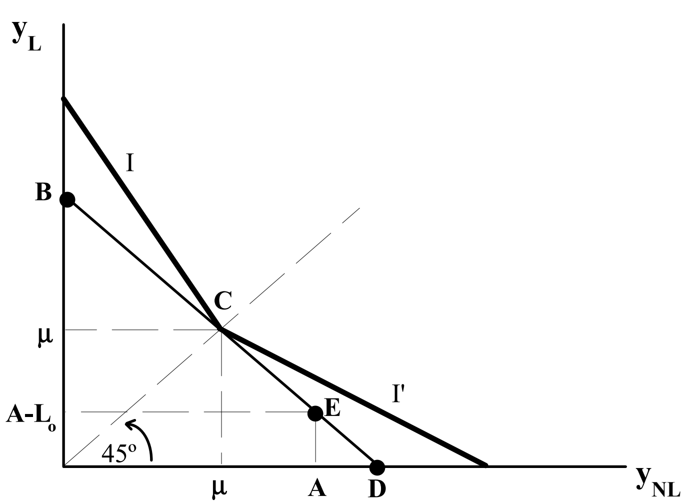

Figure 6 Full insurance at a fair price (Yaari's model)

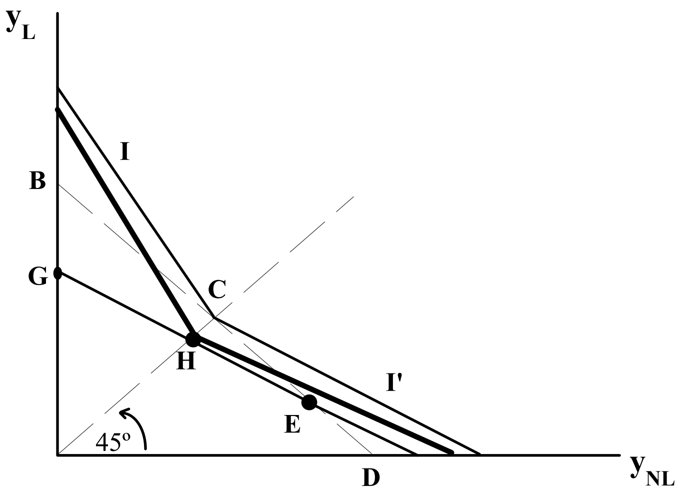

Figure 7 Full insurance at a loaded price (Yaari's model)

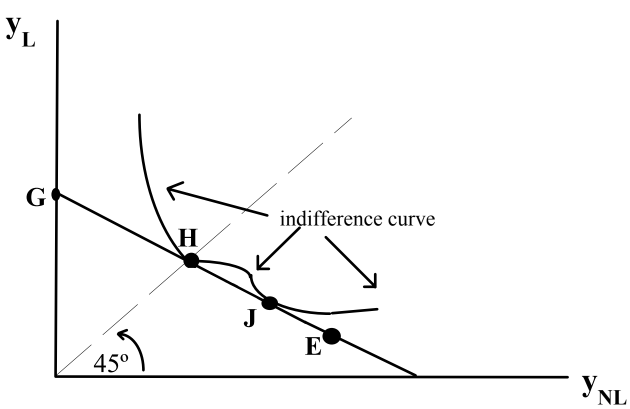

Figure 8 Both full and partial optimal insurance at a loaded price

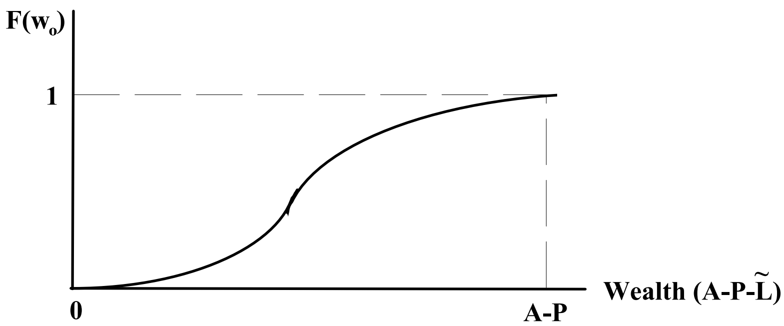

Figure 9 Wealth distribution after paying premium and experiencing a loss, but prior to insurance indemnification

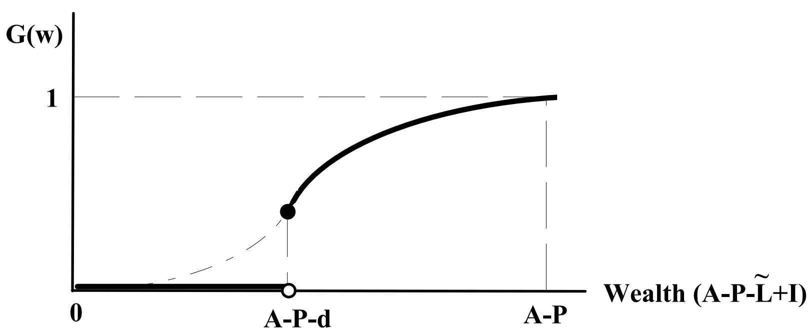

Figure 10 Final wealth distribution (deductible insurance)